Math 530

Exam 1

1. (25 pts) Suppose $f(x+iy) = u(x,y) + iv(x,y)$ is analytic on a domain Ω and $u^3 + v^3$ is a nonzero constant on Ω. Show that f must be constant on Ω.

2. (25 pts) Let $R > 0$ and let L denote the path that starts at $z = 0$ and follows the line from zero to $Re^{-i\pi/4}$ parameterized by $z(t) = te^{-i\pi/4}$, $0 \le t \le R$. Express the integral $\displaystyle\int_L e^{-z^2}\, dz$ as $a + bi$ where a and b are real integrals of real valued functions.

3. (25 pts) Let C_R denote the semicircle parameterized by $z(t) = Re^{it}$, $0 \le t \le \pi$. Prove that

$$\int_{C_R} \frac{z^3 + z^2 + z + 1}{z^5 + z^3 + z + 1}\, dz$$

tends to zero as $R \to \infty$.

4. (25 pts) Suppose f is analytic on $D_1(0) - \{0\}$ and there are real constants M and λ with $M > 0$ and $0 < \lambda < 1$ such that

$$|f(z)| \le \frac{M}{|z|^\lambda}$$

on $D_1(0) - \{0\}$. Prove that f must have a removable singularity at $z = 0$.
Hint: Consider $zf(z)$.

MATH 530 Exam 1 sol^{n}s

1. $u^3 + v^3 \equiv c, \quad c \neq 0.$

$$\begin{cases} \dfrac{\partial}{\partial x}: & 3u^2 u_x + 3v^2 v_x = 0 \\[2ex] \dfrac{\partial}{\partial y}: & 3u^2 u_y + 3v^2 v_y = 0 \end{cases}$$

C-R Eqns:

$$\underbrace{\begin{bmatrix} u_x & v_x \\ \underset{}{u_y} \overset{=-v_x}{} & \underset{}{v_y} \overset{=u_x}{} \end{bmatrix}}_{\substack{\text{Det must} = 0 \\ \parallel \\ u_x^2 + v_x^2}} \begin{pmatrix} u^2 \\ v^2 \end{pmatrix} = \vec{0}$$

Not the zero vector since $c \neq 0$.

$u_x^2 + v_x^2$. So $u_x \equiv 0$ and $v_x \equiv 0$.

Cauchy Riemann eqns $\Rightarrow v_y \equiv 0$ and $u_y \equiv 0$ too.

So $f' \equiv 0$. $f' \equiv 0$ on a <u>domain</u> $\Rightarrow$

$f \equiv$ const there.

2. $$\int_L e^{-z^2}\,dz = \int_0^R e^{-(te^{-i\pi/4})^2} \underbrace{\left[e^{-i\pi/4}\right]}_{z'(t)} dt$$

$$= \int_0^R e^{-t^2 \underbrace{e^{-i\pi/2}}_{-i}} e^{-i\pi/4}\, dt$$

$$= \int_0^R \underset{\substack{\uparrow \\ \cos t^2 + i\sin t^2}}{e^{it^2}} \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) dt$$

$$= \int_0^R \frac{\sqrt{2}}{2}\cos t^2 + \frac{\sqrt{2}}{2}\sin t^2\, dt + i\int_0^R \frac{\sqrt{2}}{2}\sin t^2 - \frac{\sqrt{2}}{2}\cos t^2\, dt$$

3. Basic poly estimates:

$$|z^3 + z^2 + z + 1| \le A|z|^3, \quad |z| > R_1$$

$$a|z|^5 \le |z^5 + z^3 + z + 1| \quad , \quad |z| > R_2$$

Basic estimate for integrals:

$$\left|\int_{C_R} f(z)\,dz\right| \le \left(\max_{C_R} |f|\right) \cdot \text{Length}(C_R)$$

$$\le \frac{AR^3}{aR^5} \cdot (\pi R) \quad \text{if } R > \text{Max}(R_1, R_2)$$

$$\longrightarrow 0 \text{ as } R \to \infty.$$

4. $|zf(z)| \le M|z|^{1-\lambda} \to 0$ as $z \to 0$.

So $F(z) = \begin{cases} zf(z) & z \neq 0 \\ 0 & z = 0 \end{cases}$ is analytic

on $D_1(0)$ by the Riemann removable singularity theorem. F has a zero of order $N \geq 1$, so $F(z) = z^N G(z)$ where G is analytic on $D_1(0)$. Finally, $f(z) = z^{N-1} G(z)$ on $D_1(0) - \{0\}$, so 0 is a removable singularity.