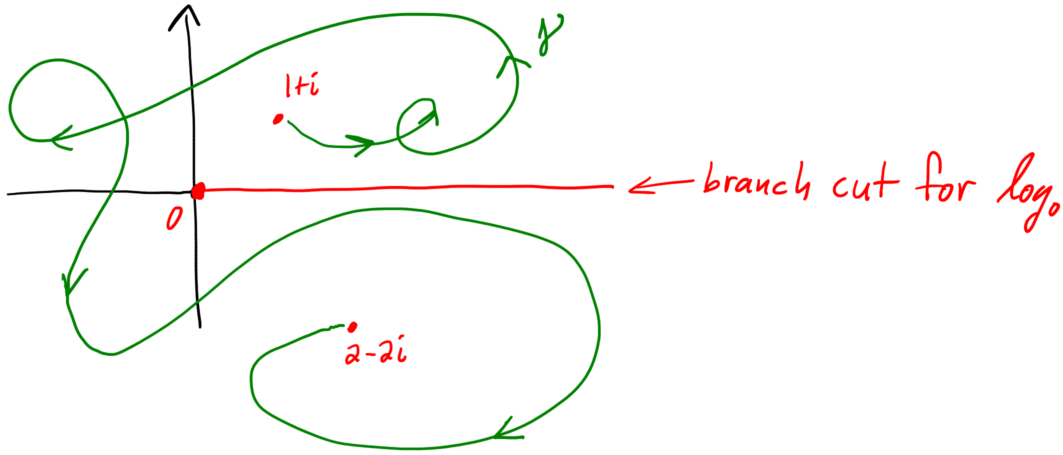
Exam 1 solutions

1. $\log_0 z = \operatorname{Ln}|z| + i\theta$, $\theta \in \arg z$ with $0 < \theta < 2\pi$.



$$\int_\gamma \frac{1}{z}\,dz = \int_\gamma \frac{d}{dz}\left(\log_0 z\right) dz = \log_0(2-2i) - \log_0(1+i)$$

$$= \left(\operatorname{Ln} 2\sqrt{2} + i\frac{7\pi}{4}\right) - \left(\operatorname{Ln}\sqrt{2} + i\frac{\pi}{4}\right)$$

$$= \underline{\operatorname{Ln} 2 + i\frac{3\pi}{2}}$$

2. a) If $u(a) < 0$ and $u(b) > 0$, then, since Ω is path connected, there is a path $\gamma : z(t)$, $\alpha \le t \le \beta$ connecting a to b in Ω. The Intermediate value theorem applied to the continuous fcn $u(z(t))$ implies that there is τ with $\alpha < \tau < \beta$ such that $u(z(\tau)) = 0$. $\lightning$

b) Suppose u is a non-vanishing $\mathcal{C}^2$-smooth harmonic fcn on $\mathbb{C}$. Then $u(z) > 0$ for all z or $u(z) < 0$ for all z.

<u>Case $u(z) > 0$, $\forall z$</u> Consider $-u$ instead and apply next case.

<u>Case $u(z) < 0$, $\forall z$</u> u has a harmonic conjugate v on $\mathbb{C}$, making $f = u + iv$ entire. e^f is also entire. But

$$|e^f| = |e^{u+iv}| = |e^u e^{iv}| = |e^u|\underbrace{|e^{iv}|}_{=1} = e^u < e^0 \text{ since } e^x \text{ strictly}$$

increasing. So $e^{f(z)}$ is a $\underline{\text{bdd}}$ entire fcn.

Liouville's $\Rightarrow$ $e^{f(z)} \equiv c$, a constant $c \in \mathbb{C}$.

Finally, $e^u = |e^f| = |c|$. So $u \equiv \ln|c|$ is constant.

3. a) $$\frac{G(z)-G(a)}{z-a} = \frac{(z-a)^2 f(z) - 0}{z-a} = (z-a)f(z)$$

and since $|f(z)| < M$, given $\varepsilon > 0$, taking $\delta = \frac{\varepsilon}{M}$ yields that

$$|(z-a)f(z)| < |z-a| \cdot M < \frac{\varepsilon}{M} \cdot \varepsilon = \varepsilon \quad \text{when } |z-a| < \delta = \frac{\varepsilon}{M},\ z \neq a.$$

Hence $\lim\limits_{z \to a} DQ$ exists and is equal to zero.

b) $$G(z) = \underbrace{G(a)}_{0} + \underbrace{G'(a)}_{0}(z-a) + a_2(z-a)^2 + \cdots \leftarrow \text{RoC} > \varepsilon$$

$$= (z-a)^2 [\underbrace{a_2 + a_3(z-a) + \cdots}_{\text{same RoC as series for } G}]$$

$$= (z-a)^2 g(z) \text{ where } g \text{ is analytic on } D_\varepsilon(a).$$

Hence $(z-a)^2 f(z) = (z-a)^2 g(z)$ when $z \in D_\varepsilon(a) - \{a\}$ and we see that $f(z) = g(z)$ when $z \in D_\varepsilon(a) - \{a\}$. So f has a removable singularity at a (by setting $f(a) = a_2 = \frac{G''(a)}{2!}$).

4. Suppose f, g analytic on a domain Ω and $fg \equiv 0$.

$\underline{\text{Case}}$ $f \equiv 0$. Done

$\underline{\text{Case}}$ $f \not\equiv 0$. Then the zeroes of f are $\underline{\text{isolated}}$ and a disc $D_\rho(z_0) \subset \Omega$ exists such that f is non-vanishing on $D_\rho(z_0)$.

Now $fg \equiv 0$ on $D_\rho(z_0) \Rightarrow g \equiv 0$ on $D_\rho(z_0)$, and the Identity theorem $\Rightarrow g \equiv 0$ on Ω. ✓

5. Suppose f is analytic on $D_1(0)$ and maps $D_1(0)$ into itself and $f(0)=0$. Then

A) $|f(z)| \le |z|$ for all $z \in D_1(0)$ and

B) $|f'(0)| \le 1$.

Furthermore, if equality holds in (A) for some $z \in D_1(0)$ with $z \ne 0$, or if equality holds in (B), then $f(z) \equiv \lambda z$ for some unimodular constant λ.