Midterm exam solutions

1. f entire.

a) Assume $|f(z)| \leq A|z|^N$, $|z| > R$.

$f =$ convergent power series $\sum_0^\infty a_n z^n$ on $\mathbb{C}$.

Cauchy estimates, Taylor's formula

$$\Rightarrow |a_n| = \left|\frac{f^{(n)}(0)}{n!}\right| \leq \frac{\max\{|f(z)| : |z| = r\}}{r^n}$$

$$\leq \frac{A r^N}{r^n} \quad \text{when } r > R, \text{ and}$$

this $\longrightarrow 0$ as $r \to \infty$ if $n > N$.

Hence $a_n = 0$ if $n > N$ and we see that f is a poly of degree N or less (because we have not shown that $a_n \neq 0$ for any $n \leq N$).

b) Assume $a|z|^N \le |f(z)|$ for $|z|>R$.

Then $|f(z)|>0$ if $|z|>R$ and consequently f has finitely many zeroes in $\mathbb{C}$. Now $\frac{1}{f}$ has finitely many poles and goes to zero at infinity. The principal parts of $\frac{1}{f}$ at the poles also go to zero at ∞. Hence $\frac{1}{f}$ minus the principal parts at the poles is an entire fcn that goes to zero at infinity, and so must be constant. We have shown that f is rational. But the only entire rational fcns are polynomials. So f is a polynomial. Finally, use the basic polynomial estimate and the given estimate to conclude that the degree of f must be $\geq N$.

<u>Alternatively</u>: $f(\frac{1}{z}) \to \infty$ as $z \to 0$. So $f(\frac{1}{z})$ has a pole at 0.

Laurent expansion:

$$f\left(\frac{1}{z}\right) = \sum_{n=1}^{M} \frac{B_n}{z^n} + \sum_{n=0}^{\infty} A_n z^n, \quad 0<|z|<r.$$

So $$f(w) = \sum_{n=1}^{M} B_n w^n + \sum_{n=0}^{\infty} \frac{A_n}{w^n}, \quad \frac{1}{r}<|w|,$$

being the Laurent expansion for

f on $|w| > \frac{1}{r}$ (by uniqueness of Laurent expansions), the negative terms must all be zero (since f is analytic on $\mathbb{C}$). So

$$f(w) = \sum_{n=1}^{M} B_n w^n + A_0$$ is a poly., etc.

2. We know that if u is harmonic on a domain Ω, then $f = u_x - i u_y$ is analytic on Ω. Note that a limit point of the set where $\nabla u = \vec{0}$ would be a limit point of the zero set of f. Hence,

if there were such a limit point, the identity theorem would imply that $f \equiv 0$ on Ω, which would yield that $\nabla u \equiv 0$ on Ω and so $u \equiv$ constant on Ω.

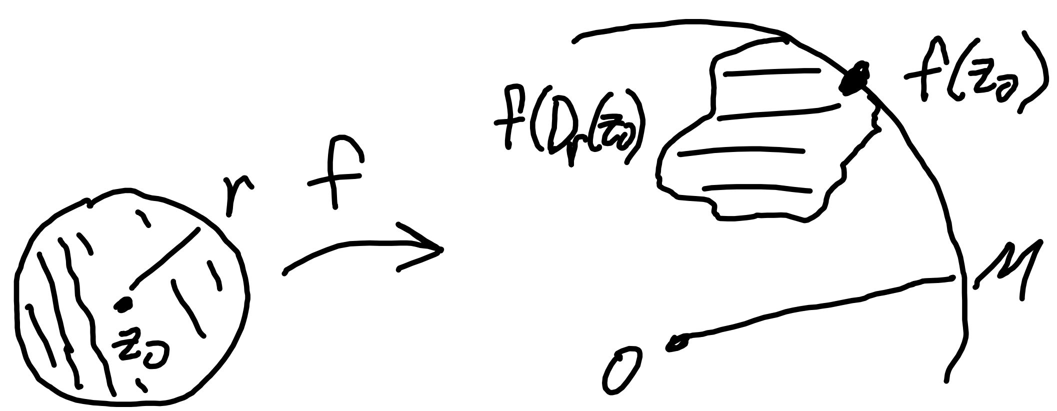
[Ω path connected: $u(B) - u(A) = \int_{\gamma_A^B} \nabla u \cdot d\vec{s}$.]

Hence, the set where the gradient of a nonconstant harmonic function on a domain vanishes cannot have a limit point.

3. Suppose f is analytic on a domain Ω and $|f(z)|$ has a local max at $z_0 \in \Omega$. Then $\exists$ disc $\overline{D_r(z_0)} \subset \Omega$ such that

$$|f(z)| \leq |f(z_0)| = M \text{ for } z \in \overline{D_r(z_0)}.$$

Picture:



If f is not constant, then O.M.T. $\Longrightarrow f(D_r(z_0))$ is open. But

$f(z_0)$ is <u>not</u> in the interior of $f(D_r(z_0))$. This contradiction yields that f must be constant on Ω. Thus, we have shown that the O.M.T $\Rightarrow$ the M.M.T.

4. $\dfrac{1}{z^2+\varepsilon z-1}$ has poles at

$-\frac{\varepsilon}{2} - \frac{\sqrt{4+\varepsilon^2}}{2}$, which is outside C,

and $-\frac{\varepsilon}{2} + \frac{\sqrt{4+\varepsilon^2}}{2}$, which is inside.

Since the zero of the denominator

of the integrand inside C is simple, the residue theorem yields

$$I = 2\pi i \left(\frac{1}{2z+\varepsilon} \right) \Bigg|_{z = -\frac{\varepsilon}{2} + \frac{\sqrt{4+\varepsilon^2}}{2}}$$

$$= \frac{2\pi i}{\sqrt{4+\varepsilon^2}}$$

5. z^4+1 has simple zeroes at $e^{in\pi/4}$, $n=1,3,5,7$. $|e^{isz}| \le 1$ in the UHP if $s \ge 0$ and in the LHP if $s \le 0$.

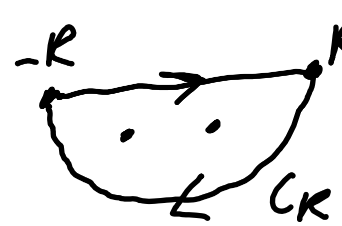
$\left| \frac{1}{z^4+1} \right| \le \frac{1}{\left| |z|^4 - 1 \right|} = \frac{1}{R^4-1}$ if $|z| = R > 1$.

For $s \geq 0$, use γ  and let

$R \to \infty$, noting that $\left| \int_{C_R} \frac{e^{isz}}{z^4+1} dz \right| \leq \frac{1}{R^4-1} (\pi R) \to 0$

as $R \to \infty$ and the residue thm to get

$$\hat{f}(s) = 2\pi i \left(\frac{e^{isz}}{4z^3} \bigg|_{z=e^{i\pi/4}} + \frac{e^{isz}}{4z^3} \bigg|_{z=e^{i3\pi/4}} \right).$$

For $s < 0$, use  to get

$$\hat{f}(s) = \underset{\substack{\uparrow \\ \text{clockwise}}}{-} 2\pi i \left(\frac{e^{isz}}{4z^3} \bigg|_{z=e^{i5\pi/4}} + \frac{e^{isz}}{4z^3} \bigg|_{z=e^{i7\pi/4}} \right).$$

These can be greatly simplified since the roots are $\pm \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$. Interesting things to note

$$\hat{f}(s) = \int_{-\infty}^{\infty} \frac{1}{t^4+1}\left(\cos st + i \sin st\right) dt$$

$$= \int_{-\infty}^{\infty} \frac{\cos st}{t^4+1}\, dt$$

and so the answers are <u>real</u>.

$$e^{ise^{i\pi/4}} = e^{is\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)} = e^{-s/\sqrt{2}}\, e^{is/\sqrt{2}}$$

this is the term that $\rightarrow 0$ rapidly as $s \rightarrow +\infty$.