Math 530 Final exam

Let $\mathcal{A}$ denote a finite set $\{a_n : n = 1, 2, \ldots, N\}$ of N distinct complex numbers.

1. Suppose that $f(z)$ is analytic on $\mathbb{C} - \mathcal{A}$ with poles at each point in $\mathcal{A}$. Prove that if the singularity of f at the point at infinity is not essential, then f must be a rational function.

2. Suppose that g is analytic on $\mathbb{C} - \mathcal{A}$. Prove that there exist constants c_n such that

$$g(z) - \sum_{n=1}^{N} \frac{c_n}{z - a_n}$$

has an analytic antiderivative on $\mathbb{C} - \mathcal{A}$.

3. Are the constants c_n in problem 2 unique? Explain.

4. Suppose that u is a real valued harmonic function on $\mathbb{C} - \mathcal{A}$. Prove that there exist real constants b_n such that

$$u(z) - \sum_{n=1}^{N} b_n \ln|z - a_n|$$

has a harmonic conjugate on $\mathbb{C} - \mathcal{A}$.

Hints: If u is harmonic, then $u_x - iu_y$ is analytic. Show that the analytic function obtained in this way from the harmonic function $\ln|z - a|$ is $(z - a)^{-1}$. Use Problem 2.

5. Suppose that u is a continuous real valued function on $\overline{D_1(0)}$ and that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + (1 - |z|)e^{i\theta})\, d\theta$$

for each $z \in D_1(0)$. This equality means that u is only known to satisfy the averaging property on internally tangent circles like the one pictured below. Prove that u is harmonic in $D_1(0)$.