Math 530 Final exam

Each problem is worth 40 points

1. Prove that Rouché's theorem implies that a complex polynomial $P(z)$ of degree $N \geq 1$ has N zeroes in the complex plane (counted with multiplicity). Consequently $P(z)$ has zeroes at points a_n of multiplicty m_n, $n = 1, \ldots, k$ where the sum of the multiplicities is equal to N. Use MA 530 results (and not algebra) to show that therefore there exists a complex constant a such that

$$P(z) = a \prod_{n=1}^{k} (z - a_n)^{m_n}.$$

2. Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

converges to a meromorphic function on $\mathbb{C}$ with double residue free poles at all the integers.

3. Let

$$H(z) = \frac{\pi^2}{\sin^2(\pi z)}.$$

a) Find the principal part of H at $z = 3$.
b) Show that

$$\lim_{y \to \pm\infty} H(a + iy) = 0$$

for each fixed real number a.

4. If $p(z)$ is polynomial of degree 2 or more, show that the sum of the residues of $1/p(z)$ on the complex plane is zero. (Hint: Use a path integral over a very large circle about the origin.) Show that $1/p(z)$ has an analytic antiderivative outside any circle containing all the zeroes of $p(z)$.

5. Suppose that u is a continuous real valued function on $\overline{D_1(0)}$ and that, for each point z_0 in $D_1(0)$, there is a circle of some radius $r < 1 - |z_0|$ centered at z_0 such that $u(z_0)$ is equal to the average of u on that circle. Prove that u is harmonic in $D_1(0)$.