Math 530 Final Exam

Each problem is worth 25points

1. State the Schwarz lemma.

2. Suppose that $f(z)$ is analytic in the unit disc $D_1(0)$ with $f(0) = 0$ and $|f(z)| < 1$ on $D_1(0)$. Prove that $\sum_{n=1}^{\infty} f(z^n)$ converges to a function that is analytic on $D_1(0)$.

3. Compute $\int_\gamma \frac{1}{z}\, dz$ where γ is a path that starts at $z = 3$ and ends at $z = -2i$ and which stays in the region $\{re^{i\theta} : r > 0 \text{ and } 0 \le \theta \le \frac{3\pi}{2}\}$.

4. Suppose $f(z)$ is analytic on the unit disc. Express the residue at $z = 0$ of the function

$$\frac{f(z)}{(e^{3z} - 1)^2}$$

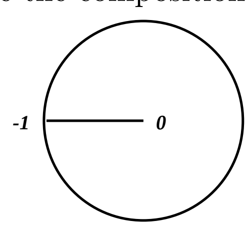
in terms of values of f and its derivatives at $z = 0$.

5. Calculate

$$\int_0^\infty \frac{1}{(x^9 + 1)}\, dx.$$

Hint: Use a contour that follows the real axis from the origin to R, then follows the circle Re^{it} from $t = 0$ to $t = 2\pi/9$, and then follows a line from $Re^{i2\pi/9}$ back to the origin. Let $R \to \infty$.

6. Let Ω denote the slit disk $D_1(0) - (-1, 0]$. Find a one-to-one conformal mapping of Ω onto the unit disc. (Express your answer as a composition of maps, but don't compute the composition.)



7. Prove that a real valued function that is harmonic on the entire complex plane must either take on every value in $\mathbb{R}$ or be constant.

8. Explain what it means for a cycle Γ to be homologous to zero in a domain Ω and give an example of such a thing for a domain that has three "holes." Prove that every closed curve in a simply connected domain is homologous to zero.