Math 530

Practice problems for the final exam

1. Suppose that ϕ is a continuous function on the unit circle in the complex plane, and let C denote the unit circle parametrized via $z(t) = e^{it}$, $0 \le t \le 2\pi$. Let $\Omega = \{w \in \mathbb{C} : |w| > 1\}$. For $w \in \Omega$, define

$$f(w) = \int_C \frac{\phi(z)}{z - w}\, dz.$$

What kind of singularity does f have at infinity? Use careful estimates and explain.

2. Suppose that A is a finite set and that $f(z)$ is analytic on $\mathbb{C} - A$ with poles at each point in A. Prove that if f has a removable singularity at infinity, then f must be a rational function.

3. Let $a_0 = 0$ and $a_1 = 1$. The Fibonacci numbers are defined inductively via

$$a_{n+2} = a_{n+1} + a_n \qquad \text{for } n = 0, 1, 2, \ldots$$

Find the radius of convergence of the power series $\sum a_n z^n$. Hint: Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and compare the power series for $z^2 f(z)$, $zf(z)$, and $f(z)$. Find a closed form formula for f. When you get the picture, make sure everything you say is true. (For example, don't say that $f(z)$ is defined someplace until you know it is.)

4. Show that a single valued analytic branch of $\sqrt{1 - z^2}$ can be defined on $\mathbb{C} - [-1, 1]$. What are the possible values of

$$\int_\gamma \frac{dz}{\sqrt{1 - z^2}}$$

when γ is a closed curve in $\mathbb{C} - [-1, 1]$.

5. How many zeroes does the polynomial

$$z^{1998} + z + 2001$$

have in the first quadrant? Explain your answer.

6. Suppose that f is a non-vanishing analytic function on the complex plane minus the origin. Let γ denote the curve given by $z(t) = e^{it}$ where $0 \le t \le 2\pi$. Suppose that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

is divisible by 3. Prove that f has an analytic cube root on $\mathbb{C} - \{0\}$.

7. Suppose that $\{a_k\}_{k=1}^N$ is a finite sequence of distinct complex numbers and that f is analytic on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$. Prove that there exist constants c_j, $j = 1, 2, \ldots, N$, such that

$$f(z) - \sum_{k=1}^{N} \frac{c_k}{z - a_k}$$

has an analytic antiderivative on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$.

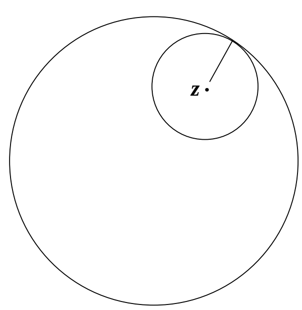
8. Suppose that f is analytic on a convex open set Ω. Prove that if the real part of f' is positive on Ω, then f is one-to-one on Ω.

9. Find a one-to-one conformal mapping of the "piece of pie" $\{re^{i\theta} : 0 < r < 1,\ 0 < \theta < \pi/4\}$ onto the horizontal strip $\{z : 0 < \operatorname{Im} z < 1\}$.

10. Suppose that u is a continuous real valued function on $\overline{D_1(0)}$ and that

$$(*) \qquad u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + (1 - |z|)e^{i\theta})\,d\theta$$

for each $z \in D_1(0)$. (This equality means that u is only known to satisfy the averaging property on circles like the one pictured below.) Prove that u is harmonic in $D_1(0)$.



Note: $(*)$ means that $u(z)$ is equal to the average of u over the internally tangent circle centered at z.