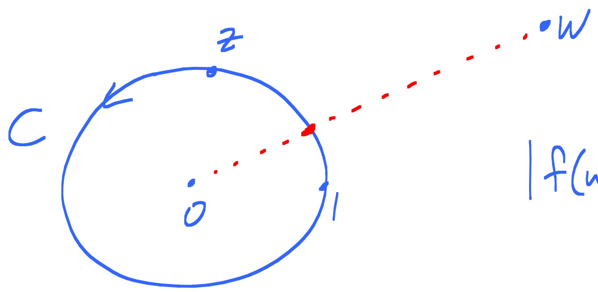
Solutions to practice problems

1. Suppose that ϕ is a continuous function on the unit circle in the complex plane, and let C denote the unit circle parametrized via $z(t) = e^{it}$, $0 \le t \le 2\pi$. Let $\Omega = \{w \in \mathbb{C} : |w| > 1\}$. For $w \in \Omega$, define

$$f(w) = \int_C \frac{\phi(z)}{z - w}\, dz.$$

What kind of singularity does f have at infinity? Use careful estimates and explain.



$$|f(w)| \le \frac{\max_C |\varphi|}{|w| - 1}\, 2\pi \cdot 1$$

$\longrightarrow 0$ as $w \to \infty$.

Type of sing of f at ∞ $\overset{\text{def}^n}{=}$ Type of sing of $f(\frac{1}{z})$ at $z=0$.

Removable sing at ∞. In fact, f has a "zero at ∞".

f analytic on $\{z : |z| > R\}$. ∞ is an "isolated sing" on $\hat{\mathbb{C}}$. 3 types: removable, pole, essential.

2. Suppose that A is a finite set and that $f(z)$ is analytic on $\mathbb{C} - A$ with poles at each point in A. Prove that if f has a removable singularity at infinity, then f must be a rational function.

Aha! $\underset{\text{value as} \to \infty.}{f} - \underbrace{\left(\sum_{1}^{N} \text{princ parts } A\right)}_{\to 0 \text{ at } \infty} = F$

F entire, bnd. Liouville's $\Rightarrow F \equiv c.$ ✓

Remark: Same thing true if f has a pole at ∞.

(If A infinite discrete set. $A = \{a_n\}_{n=1}^{\infty}$ and $a_n \to \infty$ as $n \to \infty$. ∞ is <u>not</u> an isolated sing.)

3. Let $a_0 = 0$ and $a_1 = 1$. The Fibonacci numbers are defined inductively via

$$a_{n+2} = a_{n+1} + a_n \qquad \text{for } n = 0, 1, 2, \ldots$$

Find the radius of convergence of the power series $\sum a_n z^n$. Hint: Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and compare the power series for $z^2 f(z)$, $zf(z)$, and $f(z)$. Find a closed form formula for f. When you get the picture, make sure everything you say is true. (For example, don't say that $f(z)$ is defined someplace until you know it is.)

$$z^2 f(z) = \sum_{n=0}^{\infty} a_n z^{n+2} = \sum_{n=2}^{\infty} a_{n-2} z^n$$

$$z f(z) = \ldots\ldots = \sum_{1}^{\infty} a_{n-1} z^n$$

$$f(z) = z^2 f(z) + z f(z) \quad \text{almost}$$

$-0-1\cdot z$

Suspect: $f(z)\left[1 - z - z^2\right] = z$

Radius of conv = dist (0, nearest pole)

To nail, have to go backwards from.

Define $f(z) = \dfrac{z}{1-z-z^2}$, get R. of C.

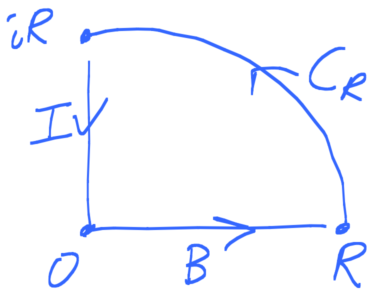
Show $(1-z-z^2)f(z) = z$ yields a_n = F. #s.

4. How many zeroes does the polynomial

$$z^{1998} + z + 2001$$

have in the first quadrant? Explain your answer.

Arg princ: $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \left(\begin{array}{l}\#\text{ zeroes } f\\ \text{inside } \gamma \text{ with mult}\end{array}\right)$



= # times $f(z)$ spins around 0 as z goes around γ in C.C. sense.

Get $\int_{C_R} \frac{f'}{f}\,dz$ by letting $R \to \infty$.

Along B: $t^{1998} + t + 2001$ sticks to $\mathbb{R}$!

$\Delta_B \arg = 0.$

Along I: $(it)^{1998} + (it) + 2001$

$-t^{1998} + it \quad 2001$

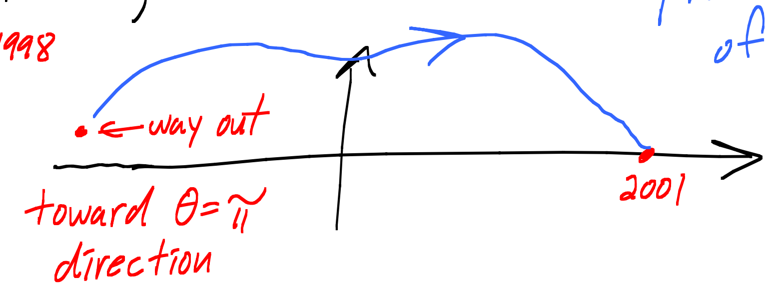
$(2001 - t^{1998}) + it$ ← in UHP!

iR : (Big negative #) $+ iR$

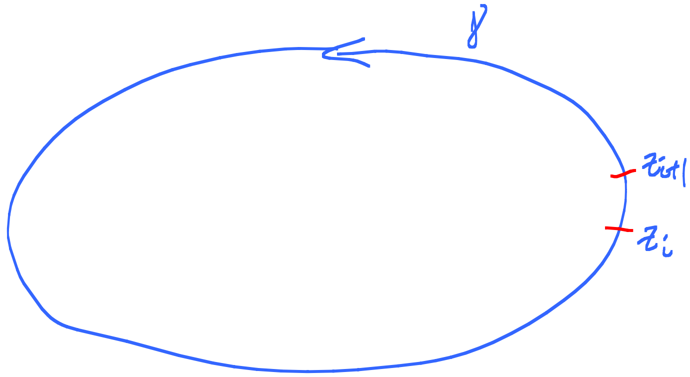
$\text{ArcTan}\,\frac{R}{2001-R^{1998}}$ $\quad 2001 - R^{1998}$

$i \cdot 0$: $\quad 2001$

$\Delta_I \arg \to -\pi$



Can use princ branch of arg.



$$\int_\gamma \frac{f'}{f}\,dz = \sum \log f(z)\Big|_{z_i}^{z_{i+1}}$$

$$= \underbrace{\sum \ln|f(z)|\Big|_{z_i}^{z_{i+1}}}_{=0} + i\,\Delta_\gamma \operatorname{Arg} f(z)$$

Important point $\quad \operatorname{Im}\left(\int_{\gamma_a^b} \frac{f'}{f}\,dz\right) = \Delta_{\gamma_a^b} \operatorname{Arg} f$

$$\int_{C_R} \underbrace{\frac{1998 z^{1997} - 1}{z^{1998} - z + 2001}}_{\sim \frac{1998}{z}}\,dz$$

$$\int_{C_R} \frac{1998}{z}\,dz = \int_0^{\pi/2} \frac{1998}{Re^{it}}\, iRe^{it}\,dt = i\,\underbrace{\frac{\pi}{2}\,1998}_{\Delta_{C_R} \arg}$$

Show $\quad \displaystyle\int_{C_R}\left(\frac{1998 z^{1997} - 1}{z^{1998} - z + 2001} - \frac{1998}{z}\right) dz \longrightarrow 0 \quad R \to \infty$

5. Suppose that f is a non-vanishing analytic function on the complex plane minus the origin. Let γ denote the curve given by $z(t) = e^{it}$ where $0 \le t \le 2\pi$. Suppose that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

is divisible by 3. Prove that f has an analytic cube root on $\mathbb{C} - \{0\}$.

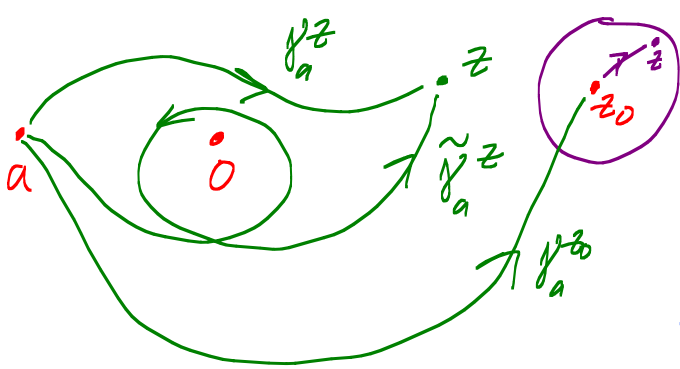
Hmmm $\quad \displaystyle\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw \leftarrow ?\ \log f$

Not well defined!

Aha! "Define" $\quad \displaystyle F(z) = \exp\left(\frac{1}{3}\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw\right)$

Step 1 $\quad F$ well defined.

$$\frac{F(z)}{\tilde F(z)} = \exp\left(\frac{1}{3}\underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde\gamma_a^z}\right)}_{\int_\gamma \ \gamma \text{ closed}} \frac{f'}{f}\,dw\right)$$



$$\frac{1}{3}\left(2\pi i \cdot N \cdot 3\right)$$

$$= e^{2\pi i N} = 1$$

Step 2 $\quad$ Is $F^3 = f$? $\quad$ Almost!

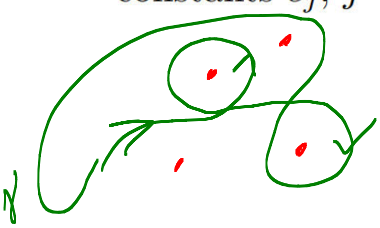
Key $\quad F$ analytic : $\quad F' = \exp\left(\frac{1}{3}\int\right)\left[0 + \frac{1}{3}\frac{f'(z)}{f(z)}\right]$

$$F' = F \cdot \frac{1}{3}\frac{f'}{f}$$

Step 3 $\quad$ Trick : $\displaystyle\frac{d}{dz}\left[\frac{F^3}{f}\right] \equiv 0.$ $\quad cF^3 = f$

Ans: $c^{1/3}F$

6. Suppose that $\{a_k\}_{k=1}^N$ is a finite sequence of distinct complex numbers and that f is analytic on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$. Prove that there exist constants c_j, $j = 1, 2, \ldots, N$, such that



$$f(z) - \sum_{k=1}^{N} \frac{c_k}{z - a_k}$$

$\Omega = \mathbb{C} - \{a_n\}_{n=1}^N$

has an analytic antiderivative on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$.

Key: analytic f has an antiderivative on a domain $\Longleftrightarrow$ $\int_\gamma f\,dz = 0$ for all closed γ in Ω.

Hmmm. $\int_\gamma f \, dz = 2\pi i \sum \mathrm{Ind}_\gamma(a_n)\,\mathrm{Res}_{a_n} f$

Gen Res thm.

F

$= 0$

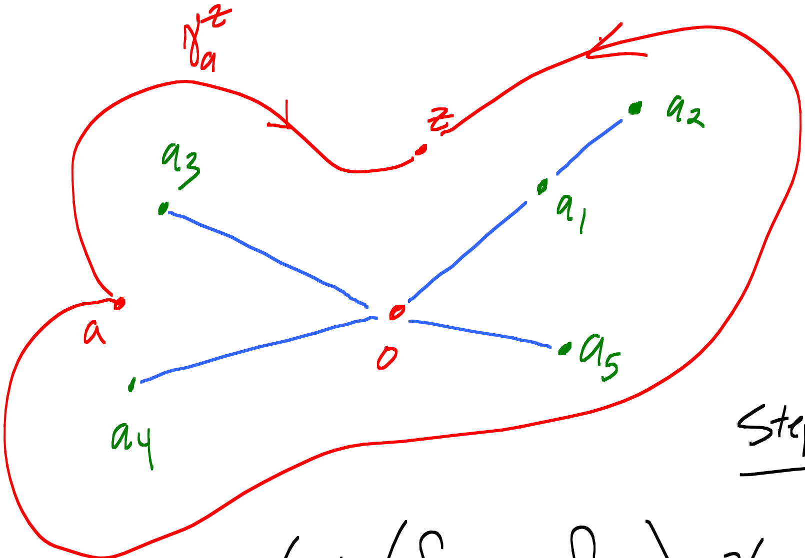
$F = f - \sum \frac{c_k}{z-a_k}$

$c_k = \mathrm{Res}_{a_k} f$

So $\mathrm{Res}_{a_k} F = 0$

7. Suppose $a_1, a_2, \ldots, a_N$ are distinct nonzero complex numbers and let Ω denote the domain obtained from $\mathbb{C}$ by removing each of the closed line segments joining a_k to the origin, $k = 1, \ldots, N$. Prove that there is an analytic function f on Ω such that

$$f(z)^N = \prod_{k=1}^{N} (z - a_k). \;= p(z)$$



Try

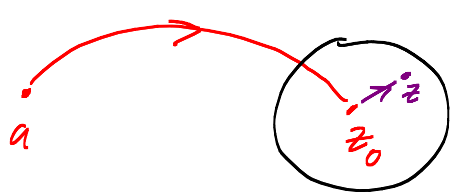
$$F(z) = \exp\left(\frac{1}{N} \int_{\gamma_a^z} \frac{p'(w)}{p(w)}\, dw\right)$$

<u>Step 1</u> $\quad \dfrac{F(z)}{\tilde{F}(z)} =$

$$\exp\left(\frac{1}{N} \underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z}\right)}_{\gamma \text{ closed}} \frac{p'}{p}\, dw\right)$$

$$= \exp\left(\frac{1}{N}\, 2\pi i \sum_1^N \underbrace{\mathrm{Ind}_\gamma(a_k)}_{\substack{\text{all same!} \\ n}}\ \underbrace{\mathrm{Res}_{a_k} \frac{p'}{p}}_{\substack{= \text{ order of simple zero} \\ = 1}}\right)$$

$$= \exp\left(\frac{1}{N}\, 2\pi i \cdot n \cdot N\right) = e^{2\pi n i} = 1 \ \checkmark$$

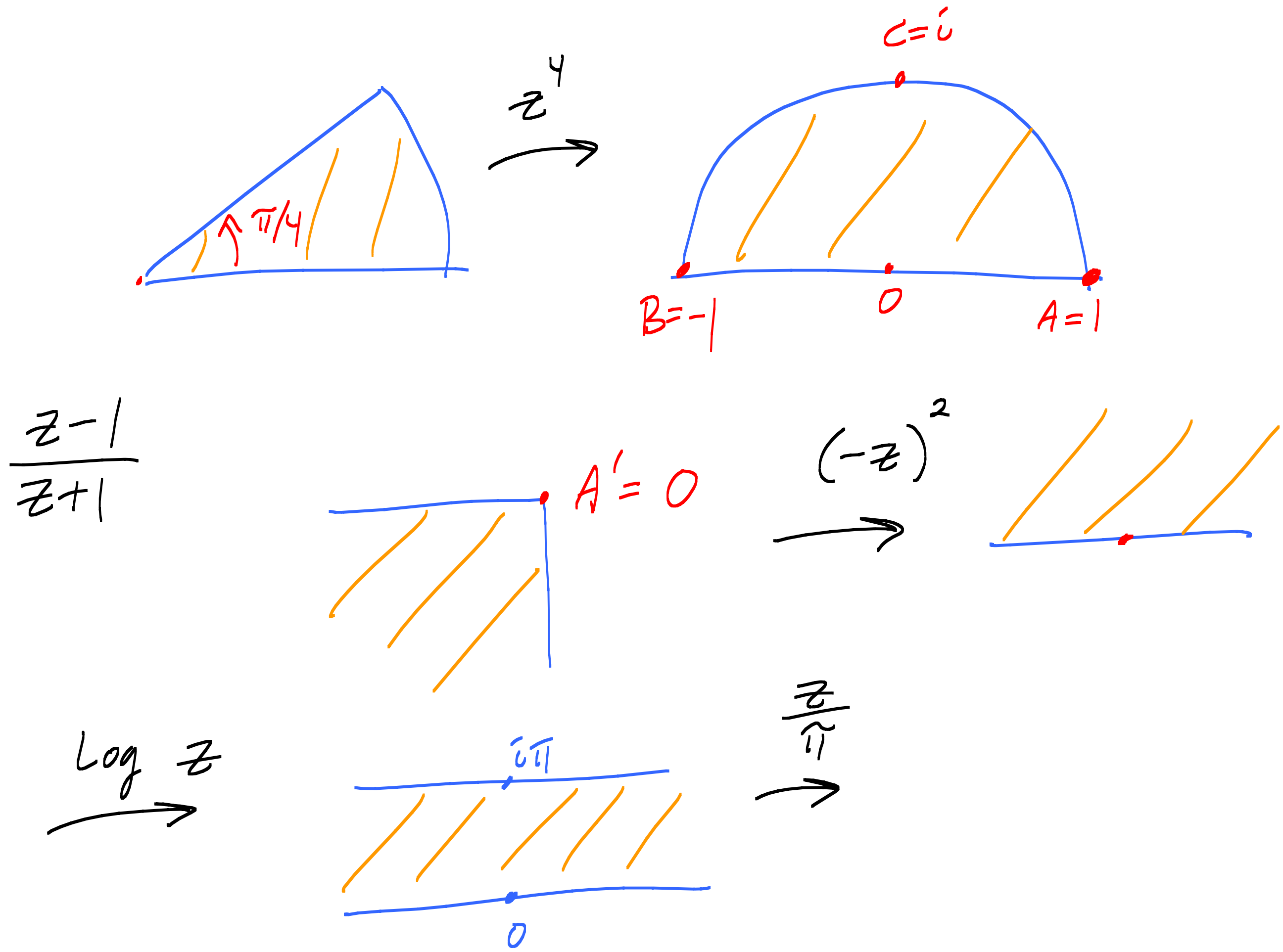


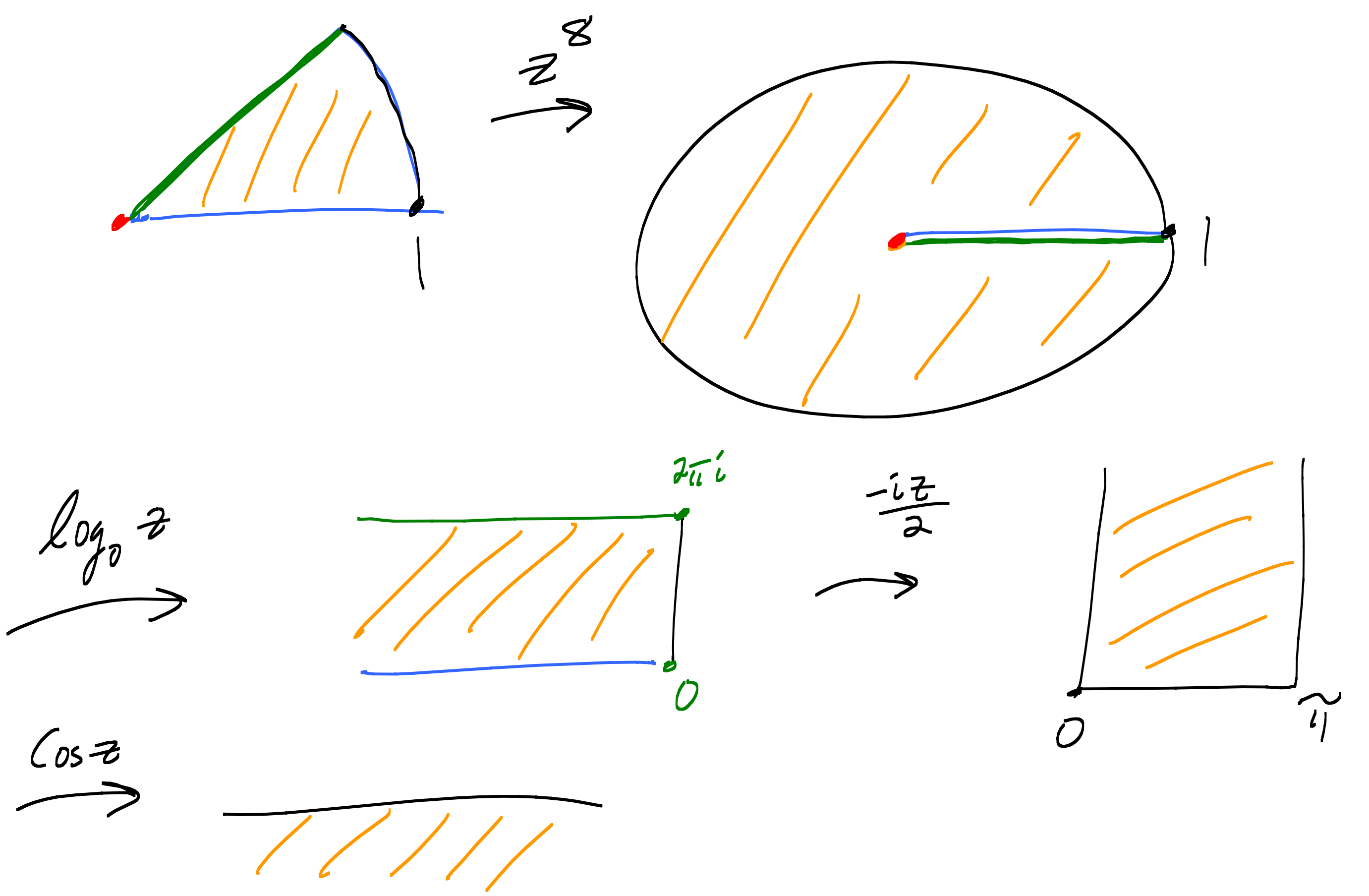
$$F' = \exp\left(\tfrac{1}{N}\int\right)\left[0 + \frac{1}{N}\frac{p'(z)}{p(z)}\right]$$

$$F' = F\, \frac{1}{N}\, \frac{p'}{p}$$

$$\frac{d}{dz}\left(\frac{F^N}{p}\right) \equiv 0, \text{ etc.}$$

8. Find a one-to-one conformal mapping of the "piece of pie" $\{re^{i\theta} : 0 < r < 1,\ 0 < \theta < \pi/4\}$ onto the horizontal strip $\{z : 0 < \operatorname{Im} z < 1\}$.



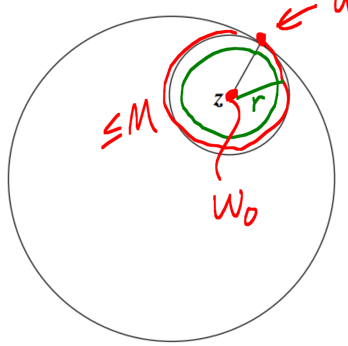


9. Suppose that u is a continuous real valued function on $\overline{D_1(0)}$ and that

$$(*) \qquad u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + (1 - |z|)e^{i\theta})\, d\theta$$

for each $z \in D_1(0)$. (This equality means that u is only known to satisfy the averaging property on circles like the one pictured below.) Prove that u is harmonic in $D_1(0)$.

$\tilde{u} - u = 0$ here!



$r \nearrow (1-|z|)$
unif cont. $\Longrightarrow$
$\tilde{u}$ satisfies $(*)$ too.

Let $\tilde{u}$ = Poisson integral of $u(e^{i\theta})$.

$\tilde{u} - u \not\equiv 0$, then $\tilde{u}(z_0) - u(z_0) = c \neq 0$. Suppose $c > 0$.

$\tilde{u} - u$ has max M on $\overline{D_1(0)}$ at some pt. $w_0 \in D_1(0)$.

$\left(\text{Ave of } \underbrace{\tilde{u} - u}_{\substack{< M \\ \text{near edge}}} \text{ on circle}\right) = M$ $\lightning$

Case $c < 0$: Apply to $u - \tilde{u}$ instead. Same $\lightning$.

So $\tilde{u} - u \equiv 0$. $\quad u = \tilde{u}$ is harmonic.