Math 530

Homework 6

1. Suppose that $P(z)$ is a complex polynomial of degree $N \geq 1$. Recall that the basic polynomial estimate says that there is a radius $R > 0$ and real constants $0 < A < B$ such that

$$A|z|^N \leq |P(z)| \leq B|z|^N$$

when $|z| > R$. Use this estimate to show that there is a radius $R > 0$ and a real constant $C > 0$ such that

$$\left| \frac{P'(z)}{P(z)} - \frac{N}{z} \right| \leq \frac{C}{|z|^2}$$

when $|z| > R$. Use this inequality and the zero counting formula

$$(\#\text{zeroes of } P \text{ inside } C_r) = \frac{1}{2\pi i} \int_{C_r} \frac{P'(z)}{P(z)}\, dz$$

from the Argument Principle to give yet another proof of the Fundamental Theorem of Algebra. (This proof will show that a polynomial of degree $N \geq 1$ has N roots, counted with multiplicity.)

2. Prove that an isolated singularity of $f(z)$ is removable as soon as either $\text{Re}\, f(z)$ or $\text{Im}\, f(z)$ is bounded above or below near the singularity.

3. Suppose that f_n is a sequence of analytic functions on a domain Ω containing $\{z : |z| \leq 1\}$ and suppose that f_n is uniformly Cauchy on the set $\{z : |z| = 1\}$. Show that f_n converges uniformly on $\{z : |z| < 1\}$ to a function f which is analytic there.

4. Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$.

5. Derive the formula

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^{2n}\theta\, d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

by integrating

$$\frac{1}{z}\left(z + \frac{1}{z}\right)^{2n}$$

around the unit circle.

6. Compute

a) $\displaystyle\int_0^\infty \frac{x^{1/3}}{1+x^2}\, dx,$ b) $\displaystyle\int_0^\infty \frac{1}{1+x^5}\, dx,$

c) $\displaystyle\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^3}\, dx,$ a real, d) $\displaystyle\int_{-\infty}^\infty \left(\frac{\sin x}{x}\right)^2 dx.$