Math 530

Homework 7

1. Find a one-to-one conformal mapping of the region common to the two disks $|z-1| < \sqrt{2}$ and $|z+1| < \sqrt{2}$ onto the unit disk.

2. Find a one-to-one conformal mapping of the region $\{z : 0 < \text{Re}\, z < 1\}$ onto the unit disk. Use the inverse of this map to show that there exists a bounded harmonic function on the unit disk whose harmonic conjugates are unbounded on the unit disk. (Remark: A similar idea can be used to show that there is a harmonic function which extends continuously to the closed disk that does not have a bounded harmonic conjugate on the disk.)

3. Let Ω denote the open set obtained by removing the interval $[-1, 1]$ from $\mathbb{C}$. Prove that there is an analytic function $F(z)$ on Ω such that $F(z)^2 = \frac{z+1}{z-1}$. *Hint:* What is the image of Ω under the map $(z+1)/(z-1)$?

4. Assume that $f(z)$ is analytic and satisfies the inequality $|f(z)-1| < 1$ in a domain Ω. Prove that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for every closed curve in Ω. How many zeroes can f have in Ω?

5. Suppose that f_n is a sequence of analytic functions on a domain Ω which converges uniformly on compact subsets of Ω to a non-constant function f. Suppose that f has a zero of order m at a point a in Ω. Prove that there is an $\epsilon > 0$ and a positive integer N such that each function $f_n(z)$ with $n > N$ has exactly m zeroes (counted with multiplicity) on $D_\epsilon(a) \subset \Omega$.

6. Suppose that f_n is a sequence of analytic functions on a domain Ω which converges uniformly on compact subsets of Ω to a function f. Suppose that $\widetilde{\Omega}$ is a domain containing $f_n(\Omega)$ for each n. Prove that, if f is not constant, then $\widetilde{\Omega}$ contains $f(\Omega)$ too.

7. Suppose that $f(z)$ has an isolated singularity at a and that there are real constants C and λ with $C > 0$ and $0 < \lambda < 1$ such that

$$|f(z)| \leq \frac{C}{|z-a|^\lambda}$$

for z in a punctured disc about a. Prove that the singularity at a is removable.