Math 530

Homework 8

1. Show that the Laurent expansion of $(e^z - 1)^{-1}$ at the origin is of the form $\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}z^{2k-1}$. The numbers B_k are known as the Bernoulli numbers. Calculate B_1 and B_2.

2. Prove that a Laurent series can be differentiated term by term. When can a Laurent series be anti-differentiated term by term?

3. Suppose that f is analytic in a neighborhood of the closed unit disc $\{z : |z| \leq 1\}$. If $|f(z)| < 1$ when $|z| = 1$, prove that f must have at least one *fixed point* in the unit disc, i.e., that there must be at least one point z with $|z| < 1$ such that $f(z) = z$.

4. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the unit disk?

5. How many roots of the equation $z^4 - 6z + 3 = 0$ fall in the annulus $\{z : 1 < |z| < 2\}$?

6. Suppose that u is a (real valued) harmonic function on a domain. Show that u cannot have any isolated zeroes.

7. If u is harmonic on a domain Ω, what can you say about the subset of Ω where the gradient of u vanishes?

8. Let $C_1(0)$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and let f be a function that is analytic on $D_r(0)$ for some $r > 1$. Prove that if $f(C_1(0)) \subset C_1(0) \setminus \{1\}$, then f is a constant function.

9. Let $\mathcal{F}$ denote the set of all analytic functions f that map the the upper half plane into the unit disc. Let $M = \sup\{|f'(i)| : f \in \mathcal{F}\}$. Show that $M < \infty$. Find all functions, if any, in $\mathcal{F}$ such that $|f'(i)| = M$.

10. Find a conformal mapping of the upper half plane minus the closed line segment joining the origin to the point i that is one-to-one and onto the unit disc.