# Math 530

Practice problems for Exam 1

**1.** What is the radius of convergence of the power series centered at zero for the function $1/(z-1-i)^{10}$?

**2.** Prove that power series can be integrated term by term. To be precise, suppose that a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R > 0$ converges on the disc $D_R(0)$ to an analytic function $f(z)$. Prove that the power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$ also has radius of convergence $R$ and that this series converges to an analytic anti-derivative of $f(z)$ inside the circle of convergence.

**3.** Suppose that $f$ and $g$ are analytic in a neighborhood of $a$. If $f$ has a simple zero at $a$, then

$$\mathrm{Res}_a \frac{g}{f} = \frac{g(a)}{f'(a)}.$$

Prove a similar formula in case $f$ has a double zero at $a$, i.e., in case $f$ is such that $f(a) = 0$, $f'(a) = 0$, but $f''(a) \neq 0$.

**4.** Consider the closed path which starts at the origin, follows the real axis to $R > 0$, then follows the circle $Re^{i\theta}$ as $\theta$ ranges from zero to $2\pi/3$, then follows the line segment joining $Re^{i2\pi/3}$ to the origin back to the origin. By letting $R \to \infty$, use this path to calculate

$$\int_0^\infty \frac{1}{1+x^3}\, dx.$$

Hint: Show that the integral over the circular part of the curve tends to zero.

**5.** Give a detailed statement and proof of the Schwarz Lemma.

**6.** Show that if $f$ is an analytic mapping of the unit disk into itself such that $f(a) = 0$, then

$$|f(z)| \leq \left| \frac{z-a}{1-\bar{a}z} \right|$$

for all $z$ in the disk.

**7.** Show that if $f$ is an analytic mapping of the unit disk into itself, then $|f'(0)| \leq 1$.

**8.** Suppose that $f$ is an analytic function on the unit disc such that $|f(z)| < 1$ for $|z| < 1$. Prove that if $f$ has a zero of order $n$ at the origin, then $|f(z)| \leq |z|^n$ for $|z| < 1$. How big can $|f^{(n)}(0)|$ be?

**9.** Suppose that $f$ is an entire function that satisfies an estimate $|f(z)| \leq C(1+|z|^N)$ for all $z$ where $C$ is a postivie constant and $N$ is a positive integer. Prove that $f$ must be a polynomial of degree $N$ or less.

**10.** Prove that if $h_1$ and $h_2$ are two analytic functions on a domain $\Omega$ such that $h_1^N \equiv h_2^N$ for some positive integer $N$, then there is an $N$-th root of unity $\lambda$ such that $h_1 = \lambda h_2$ on $\Omega$.