Suppose that $u(x,y)$ is $\mathcal{C}^1$-smooth.

Let $R(x,y)$ be defined via

$$u(x,y) = u(x_0,y_0) + u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0) + R(x,y).$$

I will show that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$

is zero.

Notice that $u(x,y) - u(x_0,y_0) =$

$$\int_0^1 \frac{d}{dt}\Big[u\big(x_0+t(x-x_0),\, y_0+t(y-y_0)\big)\Big]\,dt$$

$$= \int_0^1 \Big[u_x\big(x_0+t(x-x_0),\, y_0+t(y-y_0)\big)(x-x_0) + u_y\big(x_0+t(x-x_0),\, y_0+t(y-y_0)\big)(y-y_0)\Big]\,dt;$$

and

$u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0)$ is equal to $\int_0^1 [u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0)]\, dt.$

So $R(x,y) =$

$$\int_0^1 \left[u_x(x_0+t(x-x_0), y_0+t(y-y_0)) - u_x(x_0,y_0)\right](x-x_0)\,dt$$

$$+\int_0^1 \left[u_y(x_0+t(x-x_0), y_0+t(y-y_0)) - u_y(x_0,y_0)\right](y-y_0)\,dt.$$

Since the first partials are continuous, given an $\varepsilon > 0$, there is a $\delta > 0$ such that $|u_x(a,b) - u_x(x_0,y_0)| < \varepsilon$ if $\operatorname{dist}((a,b),(x_0,y_0)) < \delta$. Hence, if $\operatorname{dist}((x,y),(x_0,y_0)) < \delta$, then

$$|u_x(x_0+t(x-x_0), y_0+t(y-y_0)) - u_x(x_0,y_0)| < \varepsilon$$

when $0 \leq t \leq 1$. Similarly for the u_y term. Now

$$|R(x,y)| \leq \underbrace{\int_0^1 \varepsilon|x-x_0| + \varepsilon|y-y_0| \, dt}$$

$$= \varepsilon(|x-x_0| + |y-y_0|)$$

if $\operatorname{dist}((x,y),(x_0,y_0)) < \delta$. It follows that

$$\frac{|R(x,y)|}{\operatorname{dist}((x,y),(x_0,y_0))} \leq 2\varepsilon \quad \text{and}$$

this completes the proof.