MATH 693 Steve Bell

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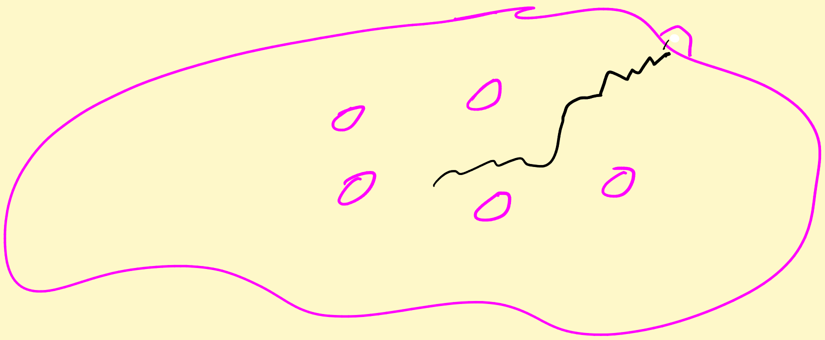


Healthy thing: go back and do easy things

The right way to think about analytic fns:

Observation: Complex polys satisfy the ave prop!

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

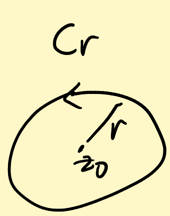
$z_0 = 0$: Easy! $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} a_k \left(re^{i\theta}\right)^k d\theta = \frac{a_k}{2\pi} r^k \int_0^{2\pi} \underbrace{\cos k\theta + i \sin k\theta}_{e^{ik\theta}}\, d\theta$

$$= 0$$

a_0 term spits out a_0 when averaged.

$z_0 \neq 0$: $\quad P(z) = P(z_0 + (z - z_0))$

$$= A_0 + A_1(z - z_0) + \cdots + A_n (z - z_0)^n$$



$C_r : z(\theta) = z_0 + re^{i\theta}$ Same calculation!

Right def^n of analytic fcn: Ω open set, f is analytic on Ω if, for any $\overline{D_r(z_0)} \subset \Omega$, f is a uniform limit of complex polys on $D_r(z_0)$.

1. so f is continuous on Ω.

2. f satisfies ave prop too.

Bergman kernel trick.

Observation #2 $\quad 1, z, z^2, z^3, \ldots$ are orthogonal on $C_1(0)$.

$$\langle u, v \rangle = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,\overline{v(e^{i\theta})}\, d\theta$$

$$z^n \overline{z^m} = e^{in\theta} \underbrace{\overline{e^{im\theta}}}_{e^{-im\theta}} = e^{i(n-m)\theta}$$

Same calculation as ave prop!

$$\langle z^n, z^m \rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

"Kernel": φ_n are $\perp$. $\quad \sum_{n=1}^{\infty} \varphi_n(z)\overline{\varphi_n(w)}$

$$K_N(z,w) = 1 + z\bar{w} + z^2\bar{w}^2 + z^3\bar{w}^3 + \ldots z^N \bar{w}^N$$

Note: $|z| < 1$, $|w| = 1$. $\quad (z^k \bar{w}^k) = (z\bar{w})^k$

Geom series $\quad K_N(z,w) = \dfrac{1-(z\bar{w})^{N+1}}{1 - z\bar{w}} \xrightarrow[N\to\infty]{} \dfrac{1}{1-z\bar{w}}$

unif on $D_\rho(0) \times C_1(0) \quad 0 < \rho < 1$.

Observation: Claim

$$P_n(z) = \int_0^{2\pi} K_N(z, e^{i\theta})\, P_n(e^{i\theta})\, d\theta$$

$\uparrow$ deg n when $N > n$.

Easy: check for z^k. Use linearity!

Can now let $N \to \infty$. Get

$$P_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1 - z(e^{-i\theta})}\, P_n(e^{i\theta})\, d\theta$$

Aha! This persists under uniform limits.

Get Cauchy Integral Formula for analytic fcns!

$$f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} - z}\, i\, e^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi i} \int_{C_1(0)} \frac{f(w)}{w - z}\, dw \quad \checkmark$$

See analytic fcns are <u>complex</u> diff'ble!