Lecture 2 Poisson & Dirichlet

Read "Something about Poisson & Dirichlet"

Last time: Define f analytic to mean f is continuous complex valued and locally (on closed discs in domain Ω) is uniform limit of complex polys $P_n(z)$.

Last time: Showed P_n satisfy ave prop $\Rightarrow$ f do too.

Also showed f satisfies the CIF.

Consequences: 1) f is complex diff'ble, in fact infinitely so.

2) $f(x+iy) = u(x,y) + i\,v(x,y)$ where u, v are C^∞-smooth and satisfy C-R eqns.

3) f = unif conv power series on closed sub discs.

Why do complex analysis? Why important?

Classic reasons: 1) Fund Thm Alg.

2) Complex exponential

3) Harmonic fcns.

My new answer: #3, the <u>averaging property</u>.

New def^n: A complex valued fcn u on a domain Ω is <u>harm-ave</u> if it is <u>continuous</u> and satisfies the ave prop:

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

when $\overline{D_r(z_0)} \subset \Omega$.

Last time: Complex polys $P(z)$ satisfy ave prop.

Hence, so do $\overline{P(z)}$.

Harm-ave Dirichlet Problem: Given continuous complex valued fcn φ on $C_1(0)$, ← unit circle

find a fcn u continuous on $\overline{D_1(0)}$, and harm-ave in $D_1(0)$, and $u = \varphi$ on $C_1(0)$.

Lemma: Given real polynomial data $\varphi(x,y)$, it's easy to write down a polynomial sol^n to harm-ave D-prob.

Pf $\varphi(x,y) = \text{polynomial in } x \text{ \& } y = p(x,y) =$

$$= p\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

$$= \mathcal{P}(z,\bar{z}) = \sum_{n,m \le N} c_{nm} z^n \bar{z}^m$$

Aha! Easy! 1 is harm-ave ✓

Case $n=m$: $z^n \bar{z}^n = (\underbrace{z\bar{z}}_{|z|^2 = 1 \text{ on } C_1(0)!})^n$

$= 1$ on $C_1(0)$, which is harm-ave. Extend terms like this to be 1.

<u>Case $n > m$</u>: $z^n \bar{z}^m = z^{n-m} \cdot \underbrace{z^m \bar{z}^m}_{=1 \text{ on } C_1(0)}$

$= z^{n-m}$ on $C_1(0)$

↑ Satisfies ave-prop! Use this extension.

<u>Case $m > n$</u>: $z^n \bar{z}^m = (\underbrace{z^n \bar{z}^n}_{=1 \text{ on } C_1(0)}) \cdot \bar{z}^{m-n}$

$= \overline{z^{m-n}}$ on $C_1(0)$

↑ Satisfies ave-prop. Use it.

See: Get harm-ave extension:

$$u(z) = P(z) + \overline{Q(z)}$$

$= \varphi$ on $C_1(0)$, harm-ave inside $C_1(0)$.

Brilliant ideas: 1) Ave prop $\Rightarrow$ Max princ.

2) Weierstraß thm $\Rightarrow$ polys are dense among continuous fcns on $C_1(0)$.

<u>Plan</u>: Given cont φ on $C_1(0)$. Get poly $p_n(x,y)$

$\xrightarrow[\text{unif}]{} \varphi$ on $C_1(0)$. Solve D-prob for p_n using $P_n(z) + \overline{Q_n(z)}$. Do these $\to u$ on $D_1(0)$?

Hmmm. Yes!

1) $p_n \xrightarrow[\text{unif}]{} \varphi$. So $\{p_n\}_{n=1}^{\infty}$ are uniformly Cauchy on $C_1(0)$. [Given $\varepsilon>0$, $\exists N$ such that

$\underbrace{|p_n - p_m| < \varepsilon}$ on $C_1(0)$ when $n,m > N$.]

2) True on $D_1(0)$ by Max princ!

See that $P_n(z)+\overline{Q_n(z)}$ are unif Cauchy on $\overline{D_1(0)}$.

3) Unif Cauchy seq on a compact set converges uniformly. So they converge unif to continuous u on $\overline{D_1(0)}$.

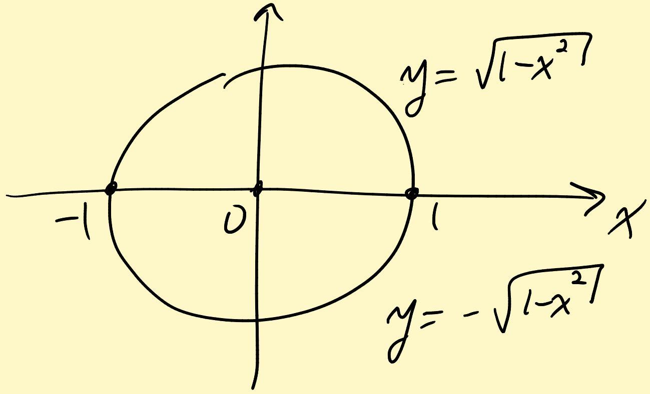
Clear that $u=\varphi$ on $C_1(0)$.

Unif conv on $D_1(0) \Rightarrow u$ satisfies ave prop!

Sol^n to harm-ave D.-prob done!

Need a little trick to get $p_n(x,y)$'s.

φ is continuous on $C_1(0)$

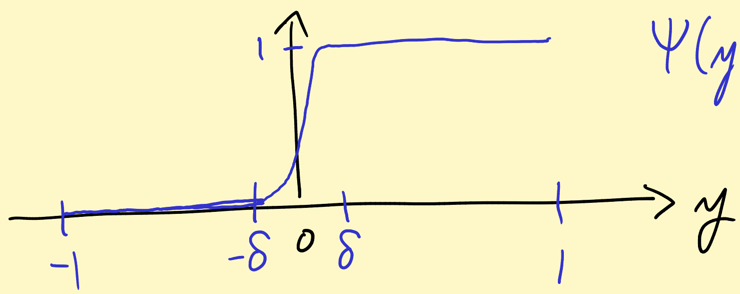


Two cont fcns on $[-1,1]$:

$$\varphi(x,\sqrt{1-x^2}) \approx p_t(x)$$
$$\varphi(x,-\sqrt{1-x^2}) \approx p_b(x)$$
polys

Can assume φ is <u>zero</u> at ± 1 by subtracting off linear thing $ax+b$.

<u>Trick</u>



$\Psi(y) \approx Q(y)$

$Q(y)\, p_t(x)$ is close to φ on top half, close to <u>zero</u> on bottom half

$[1-Q(y)]\, p_b(x)$ is close to φ on bottom, and close to zero on top.

Add them up to get $p(x,y)$ close to φ on all of $C_1(0)$.

Next time: u is given by PIF!