Lecture 3 Harmonic fcns

History: Kelvin's law: (Rate of heat flow) is proportional to temp gradient (const < 0)

Easy to deduce the Heat equ. $\frac{\partial u}{\partial t} = c \Delta u$

Steady state: $\frac{\partial u}{\partial t} \equiv 0$, u harmonic.

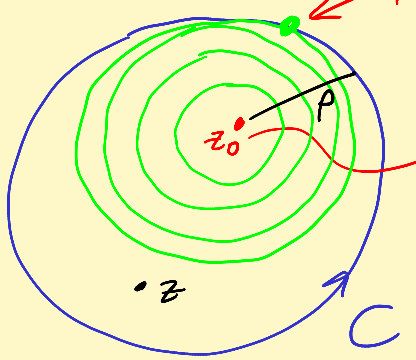
Should have had: Kelvin's steady state law: Temp satisfies the ave-property!

Today's def^n of harm-ave: u continous complex valued and satisfies the ave-prop.

Fact: Ave prop $\Rightarrow$ max princ.

Pf: Case: u real valued, continuous up to circle

Aha! $u = M$ at this bndry pt.



u max at z_0, $u(z_0) = M$

$$\underbrace{u(z_0)}_{M} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0 + \rho e^{i\theta})}_{\le M}\, d\theta$$

Conclude $u(z) \le \max_{w \in C} u(w)$.

Case: $z_0 \in C$. Done ✓ Same ineq.

Case $u = U + iV$.

$$|u(z)| \le \max_{w \in C} |u(w)| .$$

Observation Basic solutions to harm-ave D.-prob

$$1, \quad z^n, \quad \bar{z}^n \qquad n = 1, 2, \ldots$$

On $C_1(0)$: $\quad e^{im\theta} \quad : \quad m \in \mathbb{Z}$

$$\langle e^{in\theta}, e^{im\theta} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{e^{in\theta} \, \underbrace{\overline{e^{im\theta}}}_{e^{-im\theta}}}_{e^{i(n-m)\theta}} \, d\theta$$

$$= \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

Bergman kernel trick :

$$K_N(z,w) = 1 + \sum_{n=1}^{N} z^n \bar{w}^n + \sum_{n=1}^{N} \bar{z}^n w^n$$

$$= 1 + (z\bar{w}) \sum_{n=0}^{N-1} (z\bar{w})^n + (\bar{z}w) \sum_{n=0}^{N-1} (\bar{z}w)^n$$

$$= 1 + (z\bar{w}) \frac{1 - (z\bar{w})^N}{1 - z\bar{w}} + (\bar{z}w) \frac{1 - (\bar{z}w)^N}{1 - \bar{z}w}$$

$$\xrightarrow[|z|<1,\ |w|=1]{} K(z,w) = 1 + \frac{z\bar{w}}{1 - z\bar{w}} + \frac{\bar{z}w}{1 - \bar{z}w}$$

Fun calculation: $$= \frac{(1-z\bar{w})(1-\bar{z}w) + z\bar{w}(1-\bar{z}w) + \bar{z}w(1-z\bar{w})}{\underbrace{(1-z\bar{w})(1-\bar{z}w)}_{|1-z\bar{w}|^2}}$$

$w\bar{w} = |w|^2 = 1$

$$= \frac{1-|z|^2}{|1-z\bar{w}|^2} \cdot \underbrace{\left|\frac{1}{w}\right|^2}_{=1} = \frac{1-|z|^2}{|w-z|^2}$$

The Poisson kernel!

Same trick we used to get C.I.F.

$u(z) =$ Basic fcn $1, z^n, \bar{z}^n$

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} K_N(z, e^{i\theta})\, u(e^{i\theta})\, d\theta \quad \text{when } N > n.$$

Now, let $N \to \infty$.

Get $$u(z) = \frac{1}{2\pi}\int_0^{2\pi} K(z, e^{i\theta})\, u(e^{i\theta})\, d\theta$$

for basic fcns, and hence for polynomials.

Can now start with cont φ on $C_1(0)$ and use polys $p_n \rightrightarrows \varphi$ on $C_1(0)$ to get our soln u to harm-ave D. prob. Let u_n be poly sol^n with data p_n.

$$u_n(z) = \frac{1}{2\pi}\int_0^{2\pi} K(z, e^{i\theta})\, u_n(e^{i\theta})\, d\theta$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\mathcal{U}(z) = \frac{1}{2\pi}\int_0^{2\pi} K(z, e^{i\theta})\, \varphi(e^{i\theta})\, d\theta$$

Classic sol^n:

$$\mathcal{U}(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, \varphi(e^{i\theta})\, d\theta$$

Consequences: $\mathcal{U}$ is C^∞-smooth on $D_1(0)$ and $\Delta \mathcal{U} \equiv 0$ there.

Big question: Do classical harmonic fcns satisfy the ave-prop?

Def^n u is harm-PDE if

1) u is continuous

2) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ exist

3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$.

Thm harm-ave and harm-PDE are same thing!

$\Longrightarrow$ done.

$(\Longleftarrow)$: Given u harm-PDE on open nbhd of $\overline{D_1(0)}$.

Let U = Poisson integral of $u(e^{i\theta})$

Want to show $u - U \equiv 0$.

Case: Suppose not. Suppose $u-U$ has a positive value $m>0$ at some pt in $D_1(0)$. [Note: $u - U \equiv 0$ on $C_1(0)$.]

$u(z_0) = m$

Note: If $u-U$ is never positive, replace $u-U$ by $U-u$.

Clever trick $\quad \mathcal{U}_\varepsilon = u - U + \varepsilon |z|^2$ with $\varepsilon < \frac{m}{2}$

Note $\quad \mathcal{U}_\varepsilon(e^{i\theta}) = \varepsilon < \frac{m}{2} \qquad$ (A)

$\mathcal{U}_\varepsilon(z_0) \geq m \qquad$ (B)

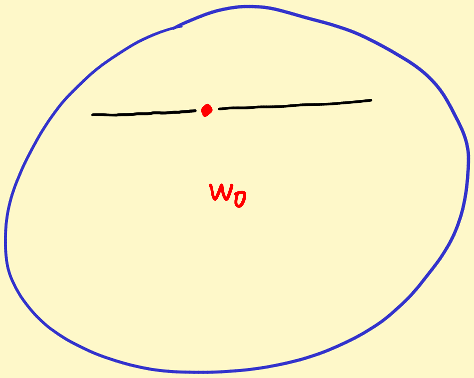
$\mathcal{U}_\varepsilon$ has a max M at a pt w_0 in $\overline{D_1(0)}$.

$w_0 \notin C_1(0)$ because of (A).

$M \geq m$.

Fiendishly clever ideas:

$\Delta u_\varepsilon > 0$ because $\Delta |z|^2 = 4$.



Think about implications of second derivative test in x-direction.

Get $\frac{\partial^2 u_\varepsilon}{\partial x^2} < 0$.

Repeat for y-dir.

Get $\frac{\partial^2 u_\varepsilon}{\partial y^2} < 0$.

See $\Delta u < 0$. $\lightning$

No class on Monday (Labor Day).