Lecture 4 The averaging property!

Last time: Almost done showing that harmonic-PDE $\Rightarrow$ harmonic-ave.

u harm-PDE : u cont, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ exist and $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \equiv 0$

(on open set Ω). Can assume u is harm-PDE on an open set containing $\overline{D_1(0)}$. Got $\mathcal{U}$ solving the harm-ave Dirichlet problem on $D_1(0)$ with continuous bndry data given by $u(e^{i\theta})$ on $C_1(0)$.

Know $\mathcal{U} =$ Poisson integral of $u(e^{i\theta})$, and so $\Delta \mathcal{U} = 0$ on $D_1(0)$.

So $\Delta(u - \mathcal{U}) \equiv 0$. And $u - \mathcal{U} \equiv 0$ on $C_1(0)$.

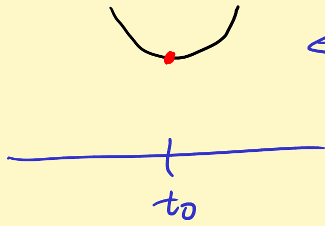
And $u - \mathcal{U}$ is continuous on $\overline{D_1(0)}$.

Last time: $u - \mathcal{U} + \underbrace{\varepsilon |z|^2}_{\Delta = 4\varepsilon > 0} = U_\varepsilon$

Cooked up $\varepsilon < \frac{m}{2}$ so that U_ε has a positive absolute max on $\overline{D_1(0)}$ inside $D_1(0)$, say at w_0.

Second deriv test (Freshman calc). u continuous, $\frac{du}{dx}, \frac{d^2u}{dx^2}$ exist on (a,b).

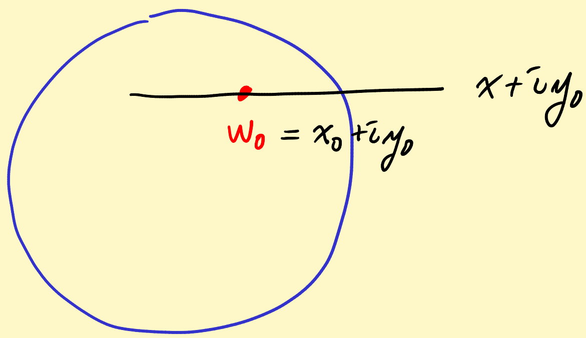
1) If $\frac{d^2u}{dx^2}(t_0) > 0$ at $t_0 \in (a,b)$, than t_0 is a local absolute min.



2) If $\frac{d^2u}{dx^2}(t_0) < 0$, local abs max. Concave down.

Pf MVT

Back to our proof.



If $\frac{\partial^2 U_\varepsilon}{\partial x^2}(x_0 + iy_0) > 0$, then w_0 is a local abs min. ↯ contradiction

So $\frac{\partial^2 U_\varepsilon}{\partial x^2}(w_0) \le 0$.

Repeat in y-direction. Get $\frac{\partial^2 U_\varepsilon}{\partial y^2} \le 0$.

Conclude $\Delta U_\varepsilon(w_0) \le 0$. But $\Delta U_\varepsilon = 4\varepsilon > 0$ at w_0. ↯

Assumption that $u - \mathcal{U}$ (or $\mathcal{U} - u$) had a positive value was wrong. So $u - \mathcal{U} \equiv 0$ on $\overline{D_1(0)}$.

$u = \mathcal{U}$

harm-PDE harm-ave

Cor Harmonic-PDE fcns are C^∞-smooth!
(Elliptic regularity.)

Importance of averaging property

Fun thing: Ave property $\Rightarrow$ Fund Thm Alg!

Pf: Suppose $P(z) = A_N z^N + A_{N-1} z^{N-1} + \cdots + A_1 z + A_0$ is a complex poly of deg $N \geq 1$ with no zeroes on $\mathbb{C}$. So $A_N \neq 0$ and $P(0) = A_0 \neq 0$.

Basic polynomial estimate There are constants a and $A > 0$ and a radius $R > 0$ such that

$$a|z|^N \leq |P(z)| \leq A|z|^N \quad \text{when } |z| > R.$$

Cor: $\left|\frac{1}{P(z)}\right| \leq \frac{1}{a|z|^N}$ when $|z| > R$.

$\frac{1}{P(z)}$ is entire and $\to 0$ as $|z| \to \infty$.

Let $A(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{P(re^{i\theta})}\, d\theta \quad \to 0$ as $r \to \infty$.

Continuity implies that $A(r) \to \frac{1}{P(0)} = \frac{1}{A_0}$ as $r \searrow 0$.

Claim: Easy to show that $A'(r) \equiv 0$.

So $A(r)$ can't $\to 0$ at ∞ since $A(0) \neq 0$. $\lightning$

$$A(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{P(re^{i\theta})} \, d\theta$$

$$A'(r) = \frac{1}{2\pi i r} \int_0^{2\pi} - \frac{P'(re^{i\theta})}{P(re^{i\theta})^2} \, i r \underbrace{e^{i\theta}}_{\frac{\partial}{\partial r}[re^{i\theta}]} \, d\theta$$

$$= \frac{1}{2\pi i r} \int_{C_r(0)} - \frac{P'(z)}{P(z)^2} \, dz$$

$$= \frac{1}{2\pi i r} \int_{C_r(0)} \frac{d}{dz} \left[\frac{1}{P(z)} \right] dz$$

$$= \frac{1}{2\pi i r} \left[\frac{1}{P(z)} \right]_{\text{START}}^{\text{END}} = 0$$

by Fund Thm Calc.

Missing piece: $\underbrace{\text{Analytic-diff}}_{\substack{\text{complex derivatives} \\ \text{exist on open set}}} \Rightarrow$ Analytic (local unif limits of complex polys)

Cauchy-Goursat lemma is still the best way to do it.

See AFTERMATH at www.math.purdue.edu/~bell/MA530 for details (Lectures 6 & 7)