Lecture 5 The $\bar\partial$-operator

In $\mathbb{C}^n$: $\bar\partial$: functions $\longrightarrow$ (0,1)-forms

$$\bar\partial u = \frac{\partial u}{\partial \bar z_1} d\bar z_1 + \frac{\partial u}{\partial \bar z_2} d\bar z_2 + \cdots + \frac{\partial u}{\partial \bar z_n} d\bar z_n$$

In $\mathbb{C}$: $\frac{\partial}{\partial \bar z}$: functions $\longrightarrow$ functions

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$\frac{\partial}{\partial \bar z}$ is the "$\bar\partial$-operator" ("dee bar")

<u>Notation</u> : $f(x+iy) = u(x,y) + i\,v(x,y)$

$$\frac{\partial f}{\partial x} = u_x + i\,v_x$$

$$\frac{\partial f}{\partial y} = u_y + i\,v_y$$

There is a $\frac{\partial}{\partial z}$-operator too : $\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$

<u>Why</u> : $\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$

When $z = x+iy$, we define complex valued 1-forms $dz = dx + i\,dy$ and $d\bar z = dx - i\,dy$

$$dx = \frac{1}{2}\left(dz + d\bar z\right), \quad dy = \frac{1}{2i}\left(dz - d\bar z\right)$$

$$\boxed{} = \underbrace{(\;\cdots\;)}_{\text{def}^n \text{ of } \frac{\partial f}{\partial z}} dz + \underbrace{(\;\cdots\;)}_{\text{def}^n \text{ of } \frac{\partial f}{\partial \bar{z}}} d\bar{z}$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

Important facts $\quad f = u + iv$

C-R eqns $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ or $\quad \frac{\partial f}{\partial \bar{z}} = 0$

1) If f is analytic, then $\frac{\partial f}{\partial \bar{z}} \equiv 0$ and $\frac{\partial f}{\partial z} = f'$.

2) If g is antiholomorphic [meaning $g(z) = \overline{h(z)}$ where h is holo]

then $\frac{\partial g}{\partial z} \equiv 0$ and $\frac{\partial g}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left[\overline{h(z)}\right] = \overline{h'(z)}$

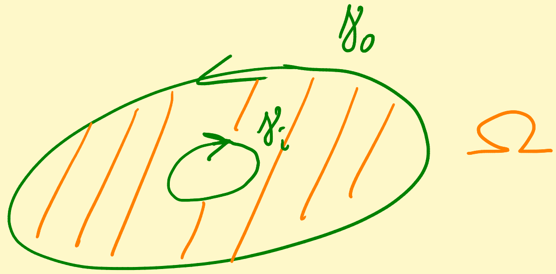
3) $h = f \circ g \;:\; h(z) = f(w)$ where $w = g(z)$.

$$\begin{cases} \dfrac{\partial h}{\partial z} = \dfrac{\partial h}{\partial w}\dfrac{\partial w}{\partial z} + \dfrac{\partial h}{\partial \bar{w}}\dfrac{\partial \bar{w}}{\partial z} \\[2ex] \dfrac{\partial h}{\partial \bar{z}} = \dfrac{\partial h}{\partial w}\dfrac{\partial w}{\partial \bar{z}} + \dfrac{\partial h}{\partial \bar{w}}\dfrac{\partial \bar{w}}{\partial \bar{z}} \end{cases}$$ complex chain rule

"Something about Poisson & Dirichlet"

Green's theorem in plane on a disc is trivial!
It is just the fund thm of calc.

Complex Green's theorem



$\Gamma = \gamma_0 \cup \gamma_i$

$$\int_\Gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \underbrace{dx \wedge dy}_{dx\,dy}$$

$dx\,dy \leftarrow$ element of area

$$\underset{\text{Stokes}}{=} \iint_\Omega d\left(P\,dx + Q\,dy\right)$$

$$= \iint_\Omega \left(\frac{\partial P}{dx}\,dx + \frac{\partial P}{\partial y}\,dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right) \wedge dy$$

Complex version

$$\int_\Gamma f\,dz = \iint_\Omega \frac{\partial f}{\partial \bar z}\,d\bar z \wedge dz = -\iint_\Omega \frac{\partial f}{d\bar z}\,dz \wedge d\bar z$$

$$\underset{\substack{\uparrow \\ \text{Stokes}}}{=} \iint_\Omega df \wedge dz = \iint_\Omega \left(\frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar z}\,d\bar z\right) \wedge dz$$

Things to check $dz \wedge dz = (dx + i\,dy) \wedge (dx + i\,dy)$ ← expand

$$= \cdots = 0$$

$$d\bar{z} \wedge d\bar{z} = 0$$

$$d\bar{z} \wedge dz = -dz \wedge d\bar{z}$$

And $$\int_{\Gamma} f \, d\bar{z} = \iint_{\Omega} \frac{\partial f}{\partial z} \, dz \wedge d\bar{z}$$

<u>Note</u> $dz \wedge d\bar{z} = (dx + i\,dy) \wedge (dx - i\,dy)$

$$= -2i \, dx \wedge dy \quad \sim -2i \, dx\,dy$$

$$= \frac{2}{i} \, dx\,dy$$

<u>Famous problem</u> "Inhomogeneous Cauchy–Riemann eqns"

Given a complex valued C^∞ fcn v on Ω,

find a complex valued u in $C^\infty(\Omega)$ with

$$\frac{\partial u}{\partial \bar{z}} = v .$$

<u>Remark</u> Possible to revise history of def^n of analytic with minimal use of Cauchy–Goursat lemma (as in "Something about P & D")

1) Use Cauchy-Goursat to get: If f is $\mathbb{C}$-diff on a domain Ω, then $\int_\Delta f\,dz = 0$ for triangles Δ in Ω.

2) Cor: $F(z) = \int_{L_a^z} f(w)\,dw$ is a complex anti-deriv of f on a convex open set. $F'(z) = f$

3) Do that twice! Get G with $G' = F$ so that $G'' = f$.

Cauchy-Rieman eqns: $G = u + iv$

$$G' = u_x + i v_x = v_y - i u_y$$

$$G'' = u_{xx} + i v_{xx} = -u_{yy} - i v_{yy}$$

$=$ $=$

See u, v are harmonic-PDE!