Lecture 6 Using $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$

Last time: Assumed f was analytic-diff.

Cauchy-Goursat $\Rightarrow \int_\Delta f\,dz = 0 \Rightarrow$ f has analytic antiderivative $F(z) = \int_{L_a^z} f(w)\,dw$ on discs.

Repeat to get G with $G' = F$. So $G'' = f$.

Assume $G = u + iv$

C-R equs: $G' = u_x + i v_x = v_y - i u_y$ (& partials exist)

repeat: $G'' = u_{xx} + i v_{xx} = -u_{yy} - i v_{yy}$

$=$ $=$

So u & v are harm-PDE $\Rightarrow$ harm-ave

So G is harm-ave

$G(z) =$ Poisson integral of $G(e^{i\theta})$

$$= \int_0^{2\pi} P(z, e^{i\theta})\, G(e^{i\theta})\, d\theta$$

$$P(z,w) = 1 + \frac{1}{1 - z\bar{w}} + \frac{1}{1 - \bar{z}w}$$

See $G(z) = c + h(z) + \overline{g(z)}$

analytic, unif limit of polys. So is g(z).

Big idea: Apply $\frac{\partial}{\partial \bar{z}}$: Get

$$0 = 0 + 0 + \overline{g'(z)}$$

C-R eqns

Conclude that $g' \equiv 0 \Rightarrow g$ is constant and

$$G(z) = (\text{const}) + h(z)$$ ← unif limit of complex polys on compact sets.

So G is analytic in any sense of the word. ✓

Oops. What about f? $\quad f = G'' = u_{xx} + i v_{xx}$

Trick: u, v harm-PDE $\Rightarrow$ harm-ave
$\Rightarrow u, v$ C^∞-smooth

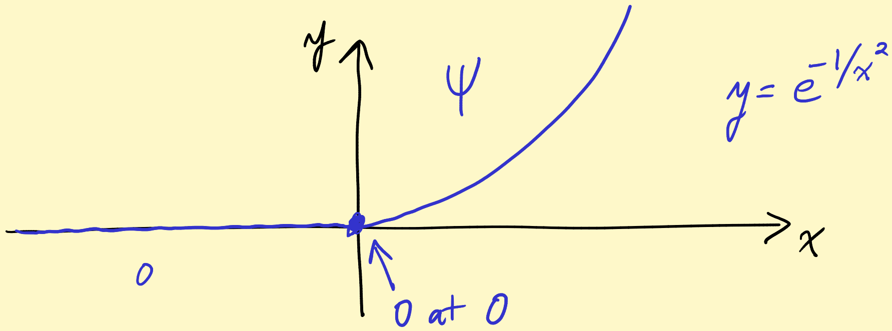
$\Rightarrow u_{xx}, v_{xx}$ are harm-PDE too! $\left[\Delta(u_{xx}) = (\Delta u)_{xx} \right]$

So f is harm-ave too.

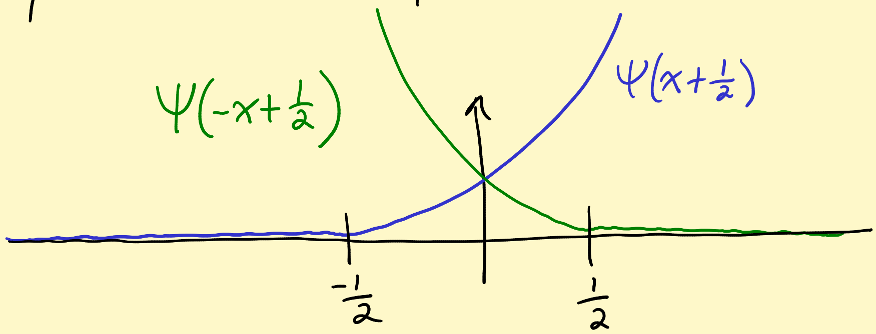
Soon: See the power of solving the $\bar{\partial}$-problem.

First: Important tool in analysis: "convolution with a C^∞-bump fcn"

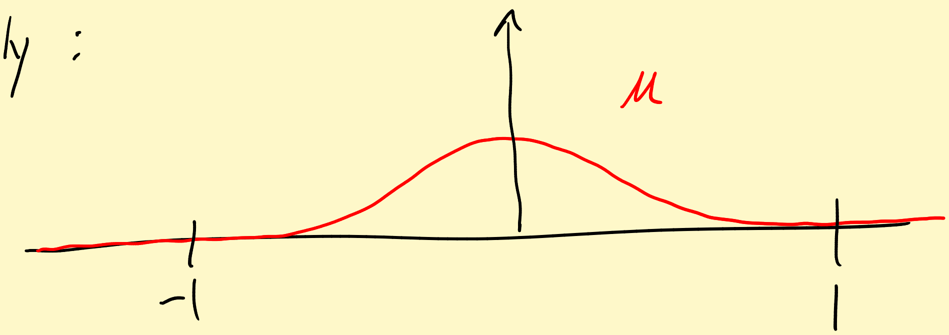
Construction:



Easy to see that ψ is C^∞-smooth



Multiply :



Bump in $\mathbb{R}^2 \approx \mathbb{C}$: $\quad U(r,\theta) = \mu(r^2) = \mu(x^2+y^2)$

$$\text{Let } c = \int_0^1 \int_0^{2\pi} U(r,\theta)\, r\,dr\,d\theta.$$

$\varphi(r,\theta) = \frac{1}{c} U(r,\theta)$ is a C^∞ radially symmetric bump fcn in $C_0^\infty(D_1(0))$.

Def^n : $\quad \varphi_\varepsilon(z) = \frac{1}{\varepsilon^2}\varphi(z/\varepsilon)$

<u>Fun facts</u>: 1) $\varphi_\varepsilon * u$ is C^∞-smooth when $u \in L^1_{loc}(\mathbb{R}^2)$.

2) If u is harm-ave, then

$$\varphi_\varepsilon * u = u \ !$$

(Another way to see harm-ave fcns C^∞-smooth!)

Today: Assume

<u>Thm</u>: Ω is a domain in $\mathbb{C}$. Given $v \in C^\infty(\Omega)$, there exists a $u \in C^\infty(\Omega)$ with $\frac{\partial u}{\partial \bar{z}} = v$.

Can use this to prove Mittag-Leffler thm on <u>any</u> domain in $\mathbb{C}$:

<u>M-L Thm</u>: Given domain Ω and seq of distinct pts $\{a_n\}_{n=1}^{\infty}$ with <u>no limit point</u> in Ω and principal parts $R_n(z) = \frac{A_{n,N}}{(z-a_n)^N} + \cdots + \frac{A_{n,1}}{z-a_n}$, there exists a meromorphic fcn h on Ω with poles precisely at the a_n having principal part $R_n(z)$ at a_n [meaning that $h - R_n$ has a

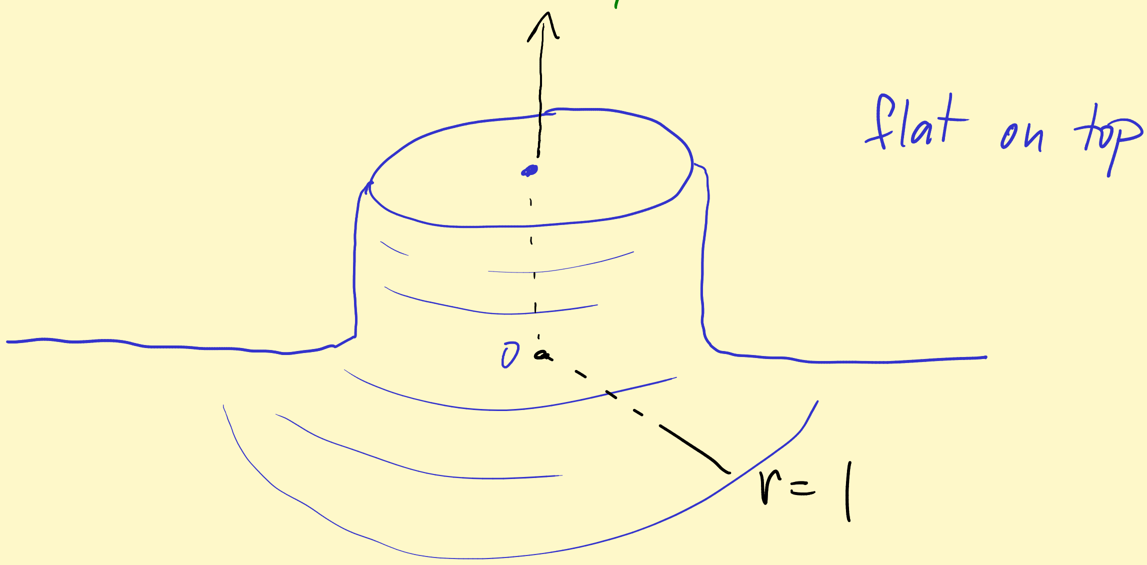
removable singularity at a_n].

Idea: Hmmm. $\sum_{n=1}^{\infty} R_n(z)$ probably won't work.

Look for a "C^∞-solution" to M-L problem.

Then "correct" it by solving a $\bar{\partial}$-problem to get an analytic sol^n.

Tool: Need a C^∞-birthday cake fcn:



$$u = \begin{cases} 1 & \text{on } D_{1/2}(0) \\ 0 & \text{elsewhere} \end{cases} \qquad \chi = \varphi_\varepsilon * u, \quad \varepsilon < \frac{1}{8}$$