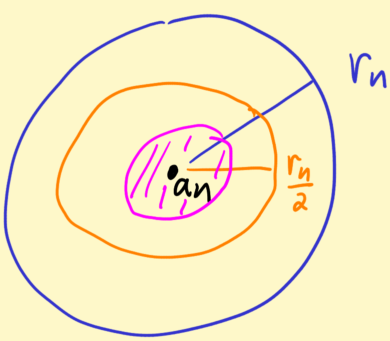
Lecture 7 $\bar\partial$ proof of the ultimate Mittag-Leffler thm

Thm: Suppose Ω is a domain and $\{a_n\}_{n=1}^{\infty}$ is a seq of distinct pts in Ω without a limit pt in Ω and $\{R_n\}_{n=1}^{\infty}$ are principal parts at the respective $\{a_n\}_{n=1}^{\infty}$. Then there exists a meromorphic h on Ω with poles precisely at the a_n with the prescibed principal parts.

Pf $\quad d_n = \inf_{m \neq n} |a_n - a_m|$ and r_n with $0 < r_n < \frac{d_n}{2}$, $\overline{D_{r_n}(a_n)} \subset \Omega$.

Discs $D_{r_n}(a_n)$ have disjoint closures.

Cook up a "C^∞-solution" to M-L problem.



Get fcn $\chi_n \in C_0^\infty(D_{r_n/2}(a_n))$ with $\chi_n \equiv 1$ on $D_{r_n/4}(a_n)$

C^∞ solution $\quad U = \sum_{n=1}^{\infty} \chi_n R_n = \begin{cases} \chi_n R_n & \text{in } D_{r_n}(a_n) \text{ each } n \\ 0 & \text{outside } \bigcup \overline{D_{r_n}(a_n)} \text{ in } \Omega \\ \text{not defined at } a_n\text{'s} & \end{cases}$

U is C^∞ on $\Omega - \{a_n\}_{n=1}^{\infty}$.

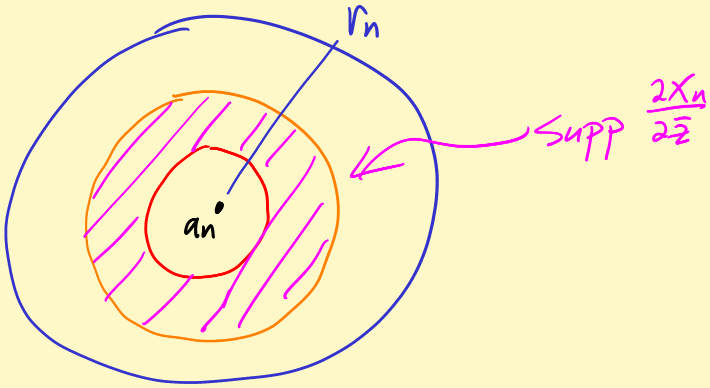
$U - R_n \equiv 0$ near a_n, so it has a "removable singularity."

Notice that $\frac{\partial U}{\partial \bar z} = \sum_{n=1}^{\infty} \underbrace{\frac{\partial}{\partial \bar z}(\chi_n R_n)}_{\frac{\partial \chi_n}{\partial \bar z} \cdot R_n + \chi_n \cdot \frac{\partial R_n}{\partial \bar z}}$ $\leftarrow \equiv 0$ on $D_{r_n}(a_n) - \{a_n\}$

$$= \sum_{n=1}^{\infty} R_n \frac{\partial \chi_n}{\partial \bar{z}} \quad \text{on } \Omega - \{a_n\}_{n=1}^{\infty}.$$

Aha! Let $v = \begin{cases} \sum_{n=1}^{\infty} R_n \frac{\partial \chi_n}{\partial \bar{z}} & \text{on } \Omega - \{a_n\}_{n=1}^{\infty} \\ 0 & \text{at } a_n\text{'s} \end{cases}$

v is a C^∞ fcn on Ω!



Now, get $u \in C^\infty(\Omega)$ with $\frac{\partial u}{\partial \bar{z}} = v$.

Aha! $h = U - u$ solves M-L problem!

1) $\frac{\partial}{\partial \bar{z}}\left(U - u\right) = \frac{\partial U}{\partial \bar{z}} - v \equiv 0$ on $\Omega - \{a_n\}_{n=1}^{\infty}$.

So h analytic on $\Omega - \{a_n\}_{n=1}^{\infty}$.

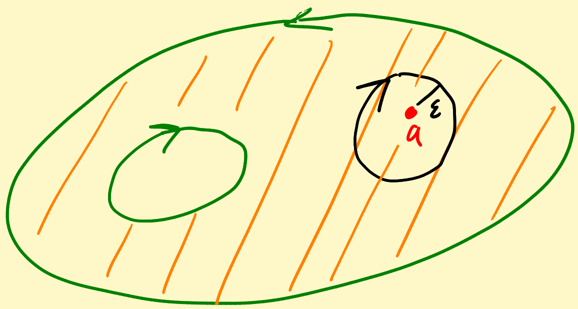
2) $h - R_n = (U - u) - R_n = \chi_n R_n - u - R_n$ on $D_{r_n}(a_n)$

$$= \underbrace{(\chi_n R_n - R_n)}_{=0 \text{ near } a_n} - \underbrace{u}_{\in C^\infty \text{ near } a_n, \text{ so bounded}}$$

Riemann Removable Singularity Thm $\Rightarrow$ a_n is removable for $h - R_n$ ✓

New improved Cauchy integral formula! (Pompeiu formula)

u C^∞-smooth



Ω

$$\Omega_\varepsilon = \Omega - \overline{D_\varepsilon(a)}$$

Green's :
$$\int_{b\Omega_\varepsilon} \frac{u(z)}{z-a}\,dz = \iint_{\Omega_\varepsilon} \underbrace{\frac{\partial}{\partial \bar z}\left[\frac{u(z)}{z-a}\right]}_{\frac{\frac{\partial u}{\partial \bar z}}{z-a}} \underbrace{d\bar z \wedge dz}_{2i\,dx\,dy}$$

$$\underbrace{\int_{b\Omega} \frac{u(z)}{z-a}\,dz}_{\text{Cauchy transform}} - \underbrace{\int_{C_\varepsilon(a)} \frac{u(z)}{z-a}\,dz}_{\text{easy to show} \to u(a)} = -\iint_{\Omega_\varepsilon} \frac{\frac{\partial u}{\partial \bar z}(z)}{z-a}\,dz \wedge d\bar z$$

<u>Claim</u> $\frac{1}{z-a}$ has an L^1_{loc} singularity at a.

Polar coord $z = a + re^{i\theta}$ near a. $\quad \frac{1}{(a+re^{i\theta})-a} = \frac{1}{re^{i\theta}}$

$$\iint_{D_r(a)/D_\varepsilon(a)} \left|\frac{1}{z-a}\right| \, dA = \int_0^{2\pi} \int_\varepsilon^r \underbrace{\frac{1}{r} \cdot r}_{=1} \, dr \, d\theta$$

Bounded as $\varepsilon \to 0$.

So $$\lim_{\varepsilon \to 0} \iint_{\Omega_\varepsilon} \frac{u_{\bar z}}{z-a} \, d\bar z \wedge dz \to \iint_{\Omega} \frac{u_{\bar z}}{z-a} \, d\bar z \wedge dz$$

$$\int_{C_\varepsilon(a)} \frac{u(z)}{z-a} \, dz = \int_0^{2\pi} \frac{u(a+\varepsilon e^{i\theta})}{(a+\varepsilon e^{i\theta})-a} \, i\varepsilon e^{i\theta} \, d\theta$$

$$= i \int_0^{2\pi} u(a+\varepsilon e^{i\theta}) \, d\theta$$

$$\to 2\pi i \, u(a)$$

$u \in C^\infty(\Omega)$:

$$\boxed{u(a) = \frac{1}{2\pi i} \int_{b\Omega} \frac{u(z)}{z-a} \, dz + \frac{1}{2\pi i} \iint_{\Omega} \frac{\frac{\partial u}{\partial \bar z}(z)}{z-a} \, dz \wedge d\bar z}$$

Hmmm. $\Omega = \mathbb{C}$. $v \in C_0^\infty(\mathbb{C})$. Want a C^∞-solution

$$\frac{\partial u}{\partial \bar z} = v \quad \text{on} \quad \mathbb{C}.$$

$$\underset{\substack{\uparrow \\ C^\infty}}{u(z)} = \underbrace{\frac{1}{2\pi i}\int_{b\Omega}\frac{u(w)}{w-z}\,dw}_{\substack{\text{analytic} \\ \text{in } z \text{ on } \Omega}} + \underbrace{\frac{1}{2\pi i}\iint_{\Omega}\frac{\frac{\partial u}{\partial \bar w}(w)}{w-z}\,dw\wedge d\bar w}_{C^\infty \text{ in } z \text{ too!}}$$

So

$$\frac{\partial u}{\partial \bar z} = 0 + \frac{\partial}{\partial \bar z}\left[\frac{1}{2\pi i}\iint_{\Omega}\frac{\overset{\text{Think } = v}{\frac{\partial u}{\partial \bar w}(w)}}{w-z}\,dw\wedge d\bar w\right]$$

<u>Hunch:</u> Given $v \in C_0^\infty(\mathbb{C})$, suspect that

$$u(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}}\frac{v(w)}{w-z}\,dw\wedge d\bar w$$

solves $\frac{\partial u}{\partial \bar z} = v$ on $\mathbb{C}$ with $u \in C^\infty(\mathbb{C})$.

It is! Pf next time.