Lecture 8 Solving the inhomogeneous Cauchy–Riemann eqns in C^∞ on $\mathbb{C}$

Thm: Given $v \in C_0^\infty(\mathbb{C})$,

$$u(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{v(w)}{w-z}\, dw \wedge d\bar{w}$$

is C^∞ smooth on $\mathbb{C}$ and satisfies $\frac{\partial u}{\partial \bar{z}} = v$ on $\mathbb{C}$.

Remark 1) If u_1 and u_2 both solve problem, then

$$\frac{\partial}{\partial \bar{z}}(u_1 - u_2) \equiv 0 \text{ on } \mathbb{C}, \text{ so } u_1 - u_2 \text{ is entire.}$$

2) u not unique. $u + (h \text{ entire})$ is another.

3) $\frac{1}{w-z}$ is locally integrable (L^1_{loc}), so easy to see $u(z)$ is C^∞ by diff under $\iint$.

e.g.
$$\underbrace{\frac{u(z+\Delta x) - u(z)}{\Delta x}}_{DQ} = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\left[\frac{v(\zeta + z + \Delta x) - v(\zeta + z)}{\Delta x}\right]}{\zeta}\, d\zeta \wedge d\bar{\zeta}$$

Oops: First change var in $\iint_{\mathbb{C}} \frac{v(w)}{\underbrace{w-z}_{\zeta}}\, dw \wedge d\bar{w}$

$$= \iint_{\mathbb{C}} \frac{v(\zeta + z)}{\zeta}\, d\zeta \wedge d\bar{\zeta}$$

$\boxed{\zeta = w - z}$

$d\zeta = dw$

$d\bar{\zeta} = d\bar{w}$

$w = \zeta + z$

$$= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{1}{\zeta} \; \underbrace{DQ}_{\substack{\downarrow \text{ unif on } \mathbb{C} \\ \frac{\partial}{\partial x} \text{ of } v(\zeta+z) \text{ and has compact support}}} \; d\zeta \wedge d\bar{\zeta}$$

L^1 (Big disc)

So $\frac{\partial u}{\partial x}$ ✓. Repeat. Same for $\frac{\partial}{\partial y}$.

$\frac{\partial}{\partial \bar{z}}$ is a linear combo of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. ✓

4) $\frac{\partial}{\partial \bar{z}} v(\zeta + z) = ?$

Facts

$$\overline{\frac{\partial u}{\partial \bar{z}}} = \frac{\partial \bar{u}}{\partial \bar{\bar{z}}} = \frac{\partial \bar{u}}{\partial z}$$

$$\overline{\frac{\partial u}{\partial z}} = \frac{\partial \bar{u}}{\partial \bar{z}}$$

Chain rule: $f(z) = v(w)$ where $w = \zeta + z$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial w}\frac{\partial w}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial \bar{z}}$$

$$= \frac{\partial v}{\partial w} \cdot 0 + \frac{\partial v}{\partial \bar{w}} \underbrace{\overline{\frac{\partial w}{\partial z}}}_{\bar{1} = 1}$$

So $\frac{\partial}{\partial \bar{z}} v(\zeta + z) = \frac{\partial v}{\partial \bar{w}}(\zeta + z)$

Pf of Thm
$$\frac{\partial u}{\partial \bar z}(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\frac{\partial v}{\partial \bar w}(\zeta+z)}{\zeta}\, d\zeta\wedge d\bar\zeta$$

Change vars back!
$$= \underbrace{\frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\frac{\partial v}{\partial \bar w}(w)}{w-z}\, dw\wedge d\bar w}_{= v(z)}$$

Aha! Piece of improved CIF. $\frac{\partial v}{\partial \bar w}$ has compact support, say in $D_R(0)$.

$= 0$ on $C_R(0)$.

CIF: $$v(z) = \frac{1}{2\pi i}\int_{C_R(0)} \frac{v(w)}{w-z}\,dw + \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\frac{\partial v}{\partial \bar w}}{w-z}\, dw\wedge d\bar w$$

Important tool: "Mittag-Leffler argument"

First M-L arg: Given distinct $a_n \in \mathbb{C}$, $a_n \to \infty$ as $n \to \infty$, $\exists$ mero fcn h on $\mathbb{C}$ with poles pricisely at a_n's with prescribed princ parts R_n at a_n.

Case Finitely many a_n: $h = \sum_{n=1}^{N} R_n$ works.

Case All $a_n \neq 0$. Then $|a_n| \neq 0$ and $|a_n| \to \infty$.

Let $r_n = \frac{|a_n|}{2}$.

$$h = \sum_{n=1}^{\infty} \Big[\underbrace{R_n(z) - p_n(z)}_{} \Big]$$

choose poly $p_n(z)$ so that

$|R_n - p_n| < \frac{1}{2^n}$ on $D_{r_n}(0)$.

(Taylor poly)

why it works: $a_m \in$ big disc $D_R(0)$.

$r_n \to \infty$, So $\exists$ N such that $r_n > R$ when $n \geq N$.

$$h = \underbrace{\sum_{n=1}^{N} \Big[R_n(z) - p_n(z) \Big]}_{\text{rational fcn with correct princ parts at } a_m \in D_R(0).} + \sum_{n=N+1}^{\infty} \Big[\underbrace{R_n(z) - p_n(z)}_{|\cdots| < \frac{1}{2^n}} \Big]$$

rational fcn

with correct princ parts

at $a_m \in D_R(0)$.

$|\cdots| < \frac{1}{2^n}$

Weier M-test

$\Rightarrow$ $\to$ analytic $g(z)$

unif on $D_R(0)$.

Conclude that $h(z) - R_m(z)$ has a removable sing at a_m. ✓

Case $a_1 = 0$. Use $R_0(z) + h$ where h solves M-L prob $\{a_n\}_{n=2}^{\infty}$.

Thm Given $v \in C^\infty(\mathbb{C})$, there is a $u \in C^\infty(\mathbb{C})$

with $\frac{\partial u}{\partial \bar{z}} = v$.

Pf: M-L arg! Let $\chi_n \in C_0^\infty(D_{n+1}(0))$

with $\chi_n \equiv 1$ on nbhd of $\overline{D_n(0)}$.

Think $$\chi_1 v + \sum_{n=1}^{N} (\chi_{n+1} - \chi_n) v = \chi_{N+1} v$$

$$\longrightarrow v \text{ as } N \to \infty.$$

Hmmm. Solve $\frac{\partial u_n}{\partial \bar z} = \chi_n v$

Think $$u = u_1 + \sum_{n=1}^{\infty} \underbrace{(u_{n+1} - u_n)}_{\text{analytic in nbhd of } \overline{D_n(0)}.\ \text{approx by } p_n!}$$

Correct $$u = u_1 + \sum_{n=1}^{\infty} \left[(u_{n+1} - u_n) - p_n \right].$$

Same idea works on $\Omega = D_1(0)$. Given $v \in C^\infty(D_1(0))$,

$\exists$ $u \in C^\infty(D_1(0))$ with $\frac{\partial u}{\partial \bar z} = v$ on $D_1(0)$.

Use $\chi_n \in C_0^\infty\left(D_{1-\frac{1}{n+1}}(0)\right)$, $\chi_n \equiv 1$ in a nbhd of $\overline{D_{1-\frac{1}{n}}(0)}$.

Need Runge thm in more general Ω.