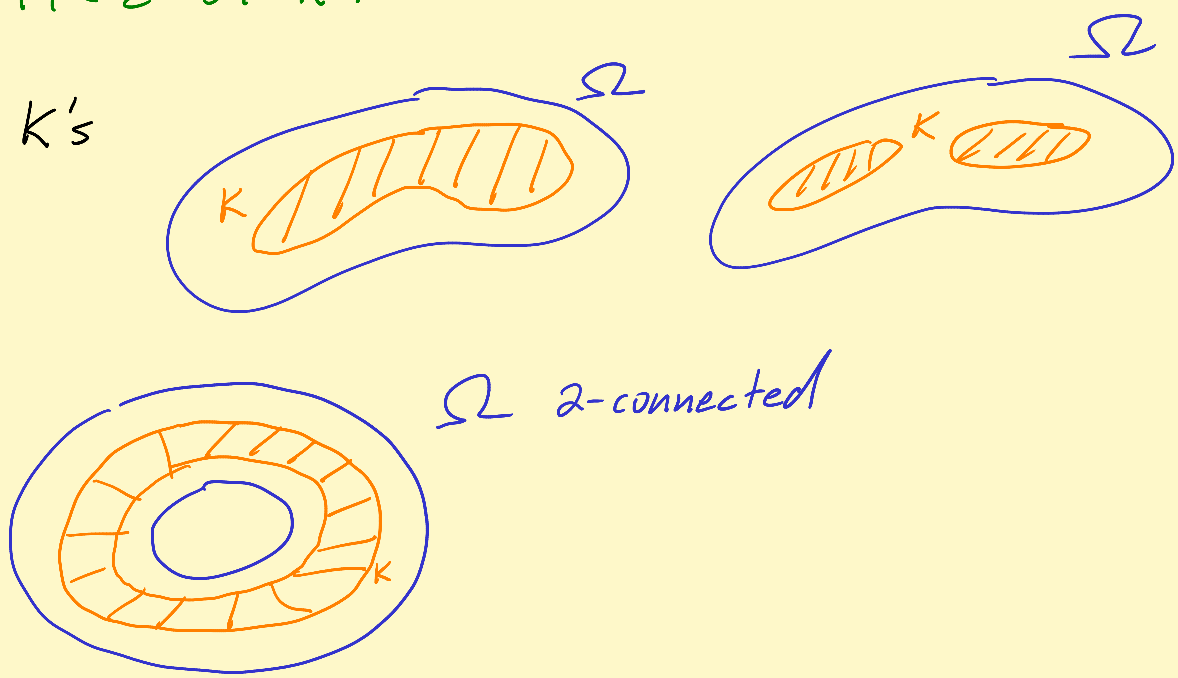
Lecture 9 Runge theorem & solving $\frac{\partial u}{\partial \bar{z}} = V$

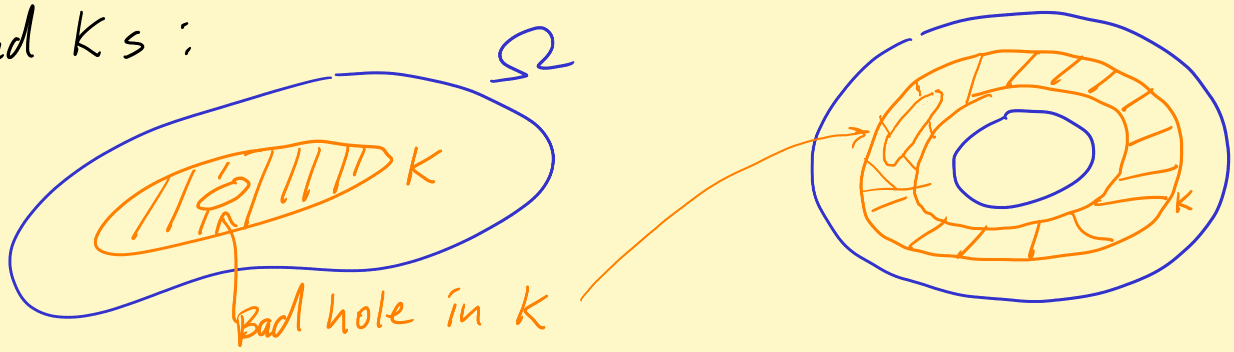
Runge thm: Suppose K is a compact set in a domain Ω, $\varepsilon > 0$ and f is analytic on an open subset of Ω containing K. Suppose that $\Omega - K$ has no open components that are relatively compact in Ω.

Then there is an analytic fcn h on Ω such that $|h - f| < \varepsilon$ on K.

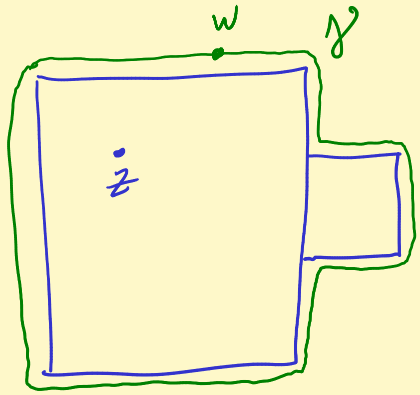
Good K's



Bad Ks:



Runge's pf. Cover K by small rectangles.
Use Cauchy integral formula:



$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}\,dw$$

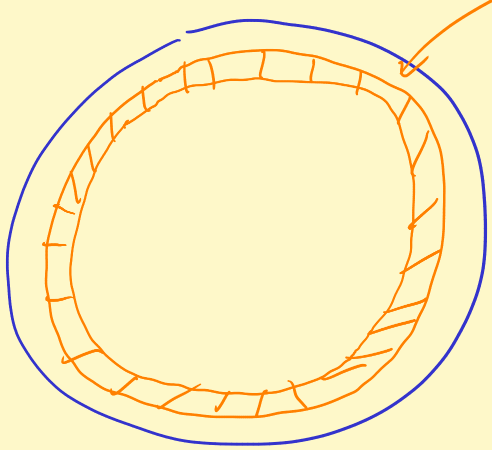
$\approx$ Riemann sum

$$\approx \sum \frac{c_j}{w_j - z}$$

Sneaky thing to do: Approximate $\frac{1}{w_j - z}$ by Laurent expansion $\sum_1^\infty \frac{A_n}{(w-z)^n}$ with w's sliding to outside of Ω. See Stein, p. 60.

<u>Solving $\frac{\partial}{\partial \bar z}$</u>

$\operatorname{supp}(\chi_{j+1} - \chi_j)V$



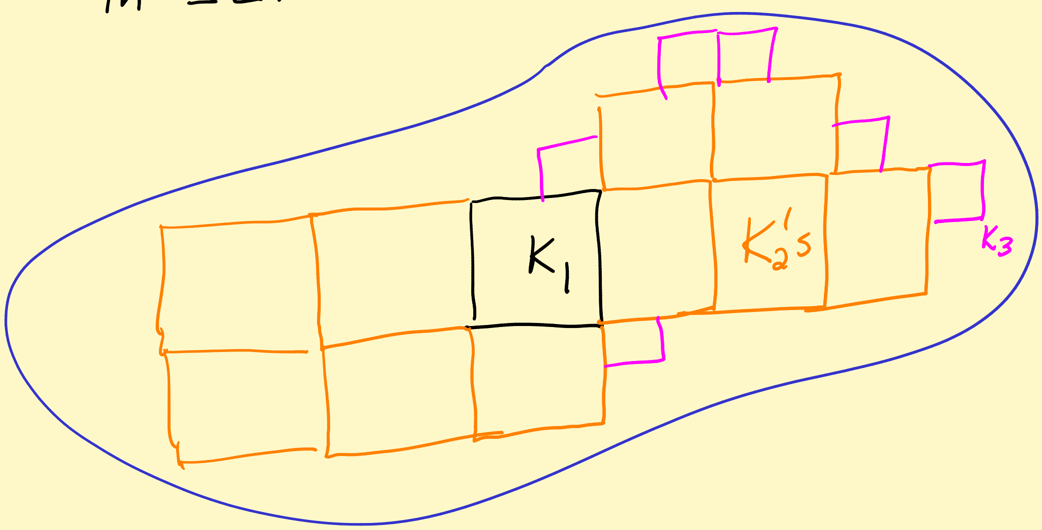
$D_1(0)$

In a general domain Ω need an "exhaustion of Ω by compact sets" — and be picky about it.

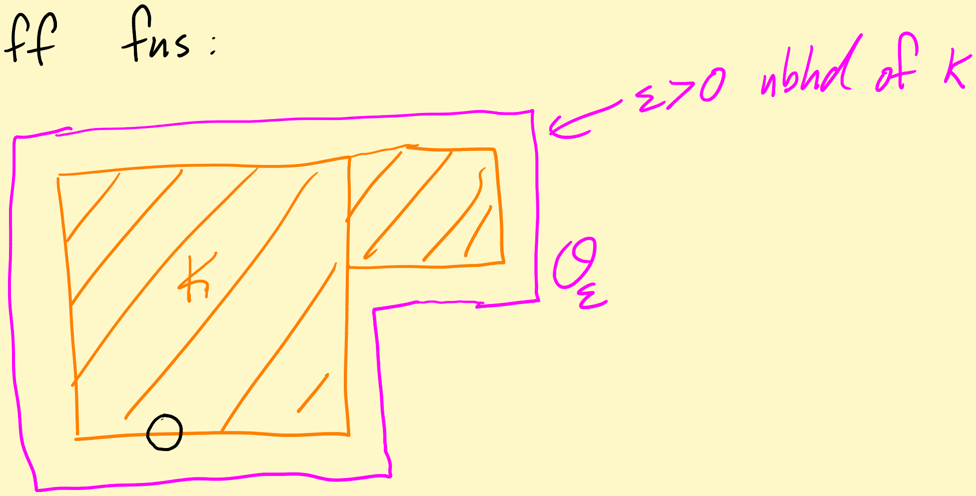
Want compact $K_1 \subset K_2 \subset \cdots \subset \Omega$

with $\bigcup_{n=1}^\infty K_n = \Omega$, <u>and</u>

$\Omega - K_n$ has no relatively compact components in Ω.



Cook up C^∞ cut-off fns:



$$\chi_K = \begin{cases} 1 & \text{on } K \\ 0 & \text{elsewhere} \end{cases}$$

Hmmm. $$\varphi_{\varepsilon/4} * \chi_{\mathcal{O}_\varepsilon} = \begin{cases} 1 & \text{on an open set containig } K \\ 0 & \text{outside } \mathcal{O}_{2\varepsilon} \end{cases}$$

Get fcns $\mathcal{X}_n \in C_0^\infty(\Omega)$, $\equiv 1$ on a nbhd of K_n and supported in K_{n+1}.

Now we can do exactly the same argument we did on unit disc to solve $\frac{\partial u}{\partial \bar z} = v$ via a M-L argument. Use Runge to get p_n's with

$$\left| (u_{n+1} - u_n) - p_n \right| < \frac{1}{2^n}.$$

See Hörmander, Chap 1 for all details.

Fun thing M-L Thm $\Longrightarrow$ Weierstraß thm in simply connected domains Ω: Suppose $\{a_n\}_{n=1}^{\infty}$ is a seq of distint pts in Ω with no limit pt in Ω, and $m_n \in \mathbb{N}$. Then there is an analytic fcn f on Ω with zeroes of order m_n's at the a_n's and no other zeroes.

[Weierstraß worked in $\Omega = \mathbb{C}$ via infinite products.]

Pf Hmmm. Suppose f works.

$$\frac{f'}{f} = \frac{m_n}{z - a_n} + (\text{analytic near } a_n)$$

Hmmm. $f(z)$ "=" $e^{\log f} = \exp\left(\int_{\gamma_a^z} \frac{f'}{f}\, dw \right)$

It works! Use M-L to get mero F on Ω with poles at only a_n with princ parts $\frac{m_n}{z-a_n}$.

"Define": $$f(z) = \exp\left(\int_{\gamma_a^z} F\, dw \right)$$

where γ_a^z is a curve starting at fixed $a \in \Omega$ and avoids $\{a_n\}_{n=1}^{\infty}$ in Ω.

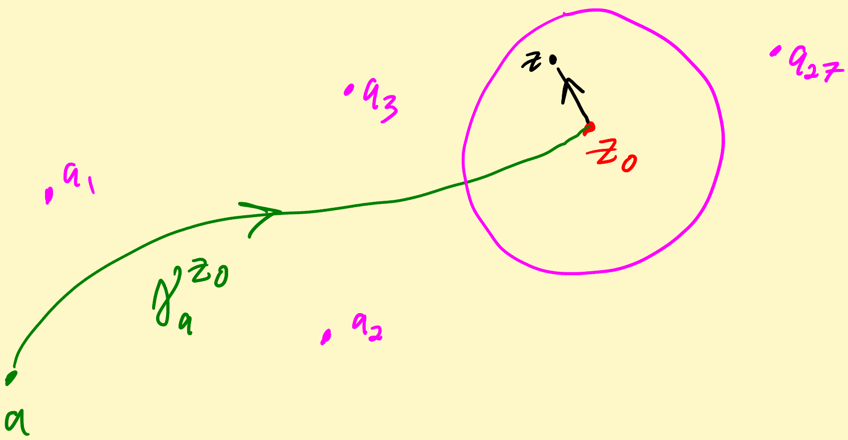
Claim f is well defined!

$$\frac{f(z)}{\tilde{f}(z)} = \exp\Bigg(\underbrace{\int_{\gamma_a^z} F\, dw - \int_{\tilde{\gamma}_a^z} F\, dw}_{\text{closed curve in } \Omega \text{ avoiding poles: } \Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)} \Bigg)$$

$$\underset{\substack{\uparrow \\ \text{Residue} \\ \text{thm}}}{=} \exp\Bigg[2\pi i \underset{\substack{\uparrow \\ \text{finite} \\ \text{sum}}}{\sum} \underbrace{\operatorname{Ind}_\Gamma(a_n)}_{\text{integer, all but finitely many} = 0} \underbrace{\operatorname{Res}_{a_n} F}_{m_n} \Bigg] = 1$$

$\Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$

<u>Claim</u> f is analytic on $\mathbb{C} - \{a_n\}_{n=1}^{\infty}$.



Lastly: Show a_n's removable and are zeroes of order m_n.