Lecture 10 The Cauchy-Kovalevskaya theorem

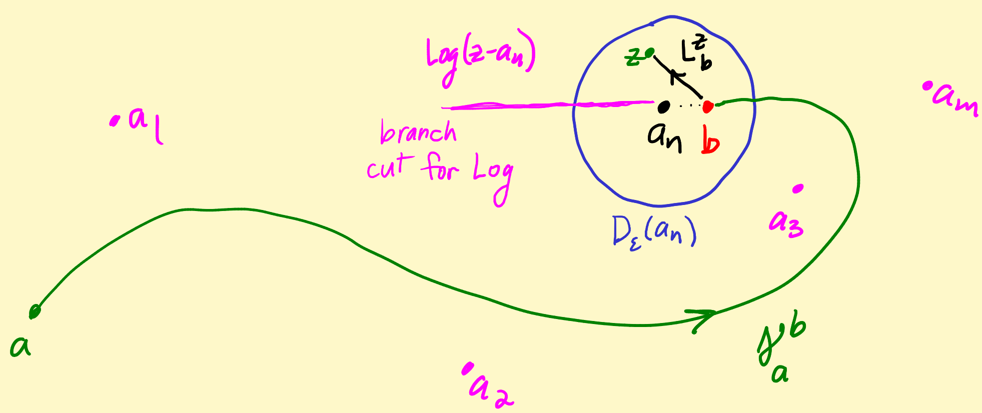
First, the end of pf of Weierstraß thm.

$$F(z) = \exp\left(\int_{\gamma_a^z} f(w)\, dw \right) \qquad f(w) = \frac{m_n}{z-a_n} + \text{analytic}$$

near a_n.

(M-L ✓)

Last step: Show a_n is a zero of order m_n.



$$\gamma_a^z = \gamma_a^b \cup L_b^z$$

$$F(z) = \exp\left(\underbrace{\int_{\gamma_a^b} f\, dw}_{\text{const}} + \underbrace{\int_{L_b^z} \frac{m_n}{w-a_n}\, dw}_{m_n \operatorname{Log}(z-a_n) - \underbrace{m_n \operatorname{Log}(b-a_n)}_{\text{const}}} + \underbrace{\int_{L_b^z} \underset{\text{on } D_\varepsilon(a_n)}{(\text{analytic})}\, dw}_{\text{analytic anti-deriv of integrand}} \right)$$

$$= \underbrace{e^{m_n \operatorname{Log}(z-a_n)}}_{(z-a_n)^{m_n}} \; e^{(\text{analytic fcn on } D_\varepsilon(a_n))} \qquad ✓$$

True on branch cut via continuity.

Def^n γ is a "real analytic curve" in $\mathbb{C}$

means $\gamma : \; z(t) = \underbrace{x(t)}_{\text{real analytic}} + i\,\underbrace{y(t)}_{\text{real analytic}} \quad t \in [a,b]$

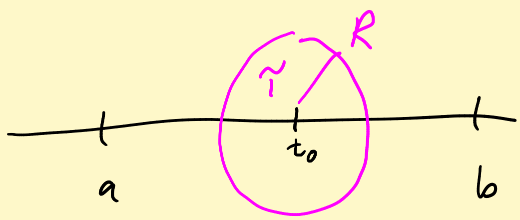
$$= z(t) = \sum_{n=0}^{\infty} A_n t^n \quad A_n \in \mathbb{C}$$

and $z'(t)$ is non-vanishing on $[a,b]$.

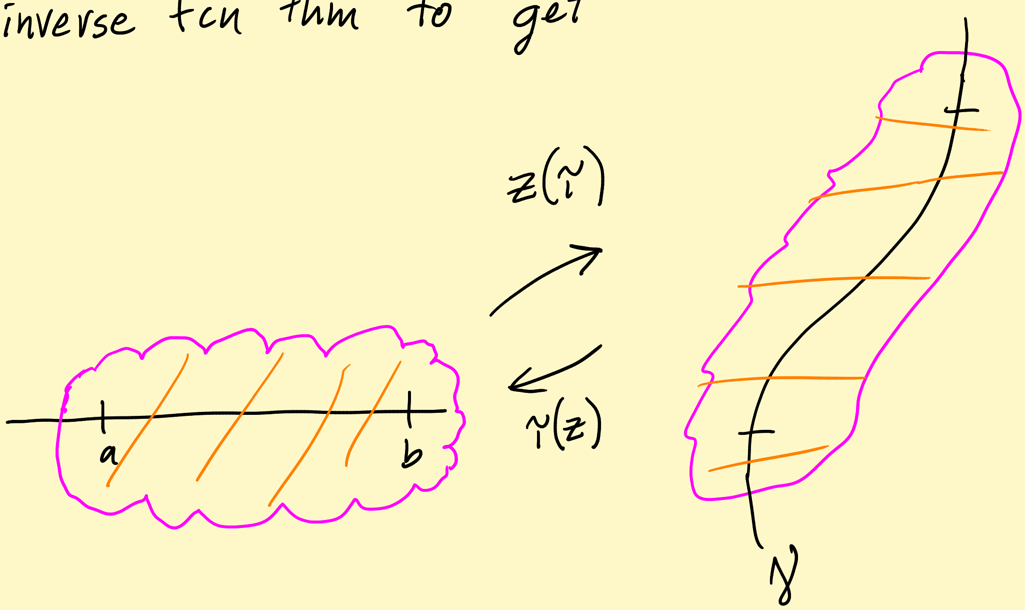
Lemma Let t go complex! Get analytic fcn

$$z(\tau) = \sum_{n=0}^{\infty} A_n(\tau - t_0)^n$$

on a "worm" about $[a,b]$.



Use inverse fcn thm to get



$z(\tau)$ analytic 1-1 onto simply conn. domain

Def^n a fcn v is r.a. in a nbhd of γ if there is an open set about γ where $v(x,y) = \sum_{n,m=0}^{\infty} A_{nm} x^n y^m$

is a r.a. fcn on open set.

<u>Lemma</u> v is r.a. on a nbhd of γ

$$\Longleftrightarrow \quad v(z(t)) \text{ is r.a. on } (a,b).$$

<u>The</u> Cauchy-Kovalevskaya thm in $\mathbb{R}^2$:

Given r.a curve γ and r.a. v near γ, there is a real analytic fcn φ in a nbhd of γ with

1) $\Delta \varphi = v$ on open set,

2) $\varphi\big|_\gamma = 0 \quad \Longleftarrow \Rightarrow \frac{\partial\varphi}{\partial n}$ parallel to $\hat{n}$

3) $\frac{\partial \varphi}{\partial n}\Big|_\gamma = 0 \quad \Longleftarrow \quad \frac{\partial\varphi}{\partial n} = \nabla\varphi \cdot \hat{n}$

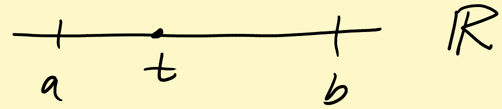
$\Rightarrow \nabla\varphi\big|_\gamma = 0.$

<u>Today's version</u> Given r.a. v near r.a. γ, there is a r.a. φ on nbhd of γ with

1) $\frac{\partial\varphi}{\partial \bar{z}} = v$

2) $\varphi\big|_\gamma = 0$

Pf: Case: γ is real line.



$$V = \sum_0^\infty A_{nm} \underset{\uparrow \atop \frac{z+\bar z}{2}}{x^n} \underset{\uparrow \atop \frac{z-\bar z}{2i}}{y^m} = \sum_{n,m=0}^\infty C_{nm} z^n \bar z^m$$

Hmmm. $\quad \varphi(z) = \displaystyle\sum_{n,m=0}^\infty \frac{C_{nm}}{m+1} z^n \bar z^{m+1}$

satisfies $\dfrac{\partial \varphi}{\partial \bar z} = V$ near real line!

Hmmm. Aha $\quad \tilde\varphi(z) = \displaystyle\sum_{n,m=0}^\infty \frac{C_{nm}}{m+1} \underbrace{z^n z^{m+1}}_{z^{n+m+1}}$ is analytic!

So $\dfrac{\partial \tilde\varphi}{\partial \bar z} = 0$.

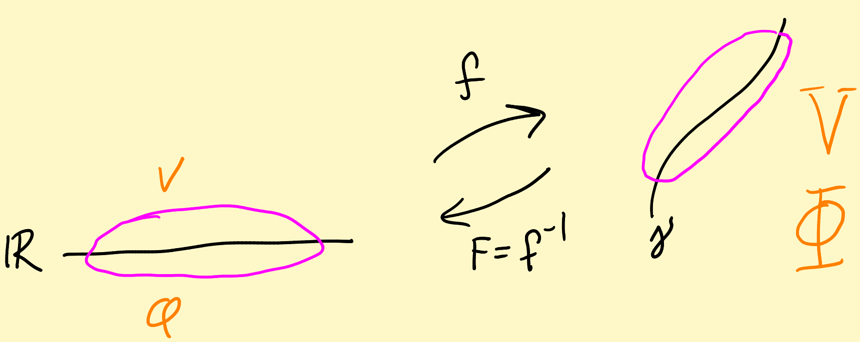
and $\quad \varphi(x) = \tilde\varphi(x) = \displaystyle\sum_{n,m=0}^\infty \frac{C_{nm}}{m+1} x^{n+m+1}$ along $\mathbb{R}$.

Sol^n to the "Cauchy problem" is

$\Phi(z) = \varphi(z) - \tilde\varphi(z)$ in $\mathbb{R}$ case.

$$\frac{\partial \Phi}{\partial \bar z} = \underset{\uparrow \atop V}{\frac{\partial \varphi}{\partial \bar z}} - 0 = V$$

Hmmm.



r.a. on open set.

Let $\Phi = \varphi \circ F$ $\qquad \varphi = \Phi \circ f$

Chain rule
$$\frac{\partial \Phi}{\partial \bar{w}} = \frac{\partial \varphi}{\partial \bar{z}} \frac{\partial \bar{F}}{\partial \bar{w}} + \frac{\partial \varphi}{\partial z} \underbrace{\frac{\partial F}{\partial \bar{w}}}_{=0,\ F \text{ analytic}}$$

$$\frac{\partial \Phi}{\partial \bar{w}} = \frac{\partial \varphi}{\partial \bar{z}}(F)\, \overline{F'(w)} \underset{\text{want}}{=} \bar{V}$$

Compose with f:
$$\frac{\partial \varphi}{\partial \bar{z}} \underbrace{\overline{F'(f(z))}}_{\frac{1}{f'(z)}} = \bar{V} \circ f$$

Aha! $\mathbb{R}$-problem is just $\frac{\partial \varphi}{\partial \bar{z}} = \underbrace{\overline{f'(z)}\, \bar{V}(f(z))}_{v}$

Get $\varphi\big|_{\mathbb{R}} = 0$, r.a., $\frac{\partial \varphi}{\partial \bar{z}} = v$. Done!

Get $\Phi = \overline{F'}\,(\varphi \circ F)$ solves Cauchy prob on γ.

Note Can also solve $\frac{\partial \varphi}{\partial z} = v$ problem:

Solve $\frac{\partial \psi}{\partial \bar{z}} = \bar{V}$, $\psi\big|_{\gamma} = 0$

Take conjugate: $\overline{\frac{\partial \psi}{\partial \bar{z}}} = \overline{\bar{V}}$: $\frac{\partial \bar{\psi}}{\partial z} = V$

$$\frac{\partial \bar{\Psi}}{\partial z} = V. \qquad \varphi = \bar{\Psi} \text{ solves } \frac{\partial \varphi}{\partial \bar{z}} = V$$

with $\varphi|_\gamma = 0$.

Important fact $\quad \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \frac{1}{4}\Delta$

Next time: Can solve original problem easily now!