Lecture 11 More on the Cauchy-Kovalevskaya theorem

Remarks: Weierstraß theorem $\Rightarrow$ every domain in $\mathbb{C}$ is a "domain of holomorphy"

e.g. Unit disc $= \Omega$. Take $a_n = r_n e^{i\theta_n}$, $r_n \nearrow 1$, $\{e^{i\theta_n}\}_{n=1}^{\infty}$ dense in $C_1(0)$. Get f with simple zeroes at a_n's.

f can't extend analytically to any bigger open set.

[If it did, it would be zero on an open arc in $C_1(0)$ $\Rightarrow f \equiv 0$. ↯]

Thm C-K Thm (classic version): γ r.a. curve, v r.a. on a nbhd of γ. $\exists$ r.a. φ on a nbhd of γ with

1) $\Delta \varphi = v$

2) $\varphi|_\gamma = 0$

3) $\nabla\varphi|_\gamma = 0$

Pf: Want $\Delta\varphi = \frac{\partial}{\partial \bar z}\Big(\underbrace{4\frac{\partial \varphi}{\partial z}}_{\psi}\Big) = v$

Step 1: Get ψ with $\frac{\partial \psi}{\partial \bar z} = v$, $\psi|_\gamma = 0$. ψ r.a.

Step 2: Get φ with $\frac{\partial \varphi}{\partial z} = \frac{1}{4}\psi$, $\varphi|_\gamma = 0$.

Aha! $\Delta\varphi = v$, φ r.a., $\varphi|_\gamma = 0$.

Claim: $\nabla\varphi|_\gamma = 0$ too.

Why: $z(t) : [a,b] \to \mathbb{C}$ parametrizes γ.

Then $\varphi(z(t)) \equiv 0$.

Diff wrt t, and use "complex chain rule" in the form

$$\frac{d}{dt}\varphi(z(t)) = \frac{\partial\varphi}{\partial z}\frac{dz}{dt} + \frac{\partial\varphi}{\partial\bar z}\frac{d\bar z}{dt} \equiv 0$$

$$\underbrace{\frac{\partial\varphi}{\partial z}}_{=\frac{1}{4}\psi \equiv 0 \text{ on } \gamma} z'(t) + \frac{\partial\varphi}{\partial\bar z}\underbrace{\overline{z'(t)}}_{\text{not zero}} \equiv 0$$

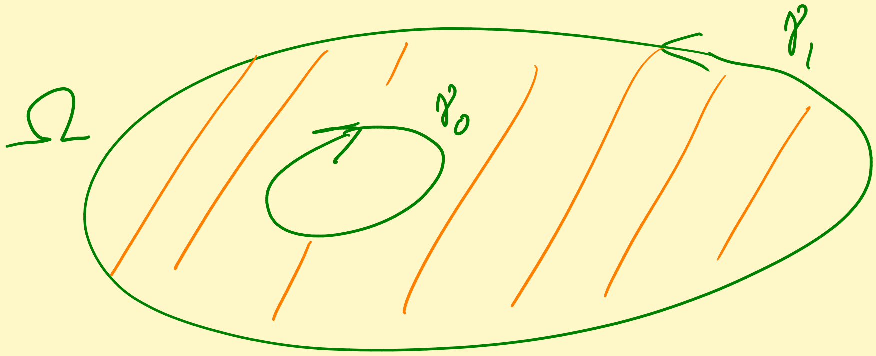
Conclude that $\frac{\partial\varphi}{\partial\bar z} \equiv 0$ on γ. Know $\frac{\partial\varphi}{\partial z} \equiv 0$ on γ too.

This implies $\nabla\varphi \equiv 0$ on γ. ✓

Why: $\begin{cases} \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{cases}$ $\quad \underline{\begin{cases} \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar z} \\ \frac{\partial}{\partial y} = \frac{1}{i}\left(\frac{\partial}{\partial \bar z} - \frac{\partial}{\partial z}\right) \end{cases}}$

$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}$ zero $\Rightarrow$ $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ zero

Domains with r.a. boundary



γ's r.a. curves.

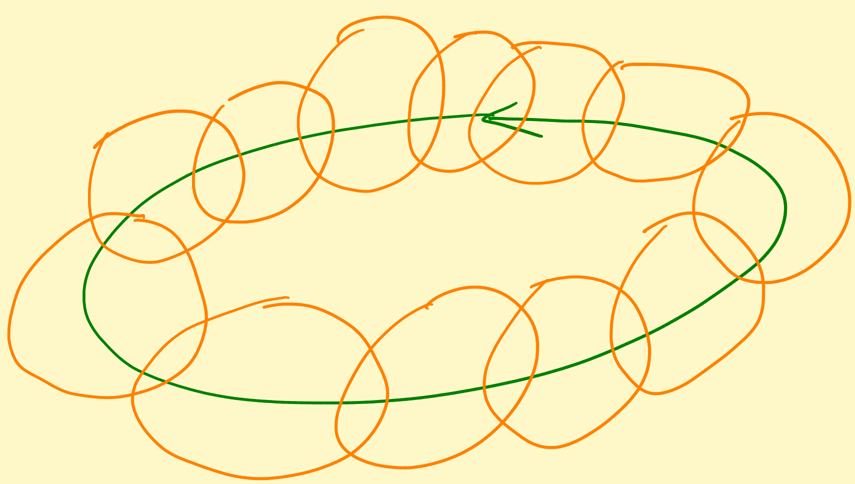
equivalent to $\Omega = \{z : r(z) < 0\}$, r real valued

where r is r.a. in a nbhd of the boundary

and $\nabla r \neq 0$ on $b\Omega$. [r is a <u>defining fcn</u> for Ω.]

ex: $r(z) = |z|^2 - 1$
$D_1(0) = \{r < 0\}$.

<u>Tool</u> C^∞ partions of unity.



Open cover of γ.
Take a finite subcover.
$\{U_n\}_{n=1}^N$

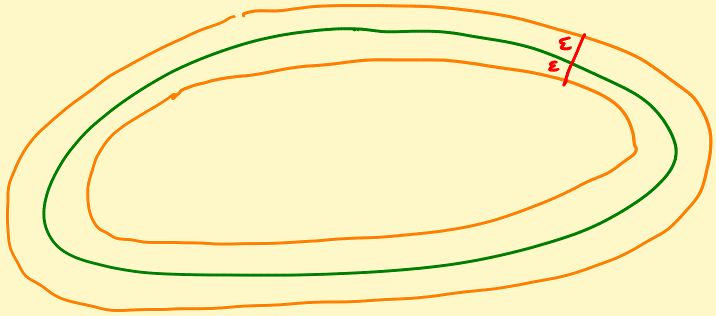
There exists a C_0^∞ partition of unity $\{\varphi_n\}_{n=1}^N$ for γ

"subordinate" to cover: $\varphi_n \in C_0^\infty(U_n)$,

meaning that $\sum_{n=1}^N \varphi_n \equiv 1$ in a nbhd of γ.

See Spivak: Calculus on manifolds.

Sketch



$\chi_\varepsilon * \varphi_{\varepsilon/4} = X_\varepsilon$

$\in C_0^\infty(\operatorname{supp} \chi_{2\varepsilon})$, $\equiv 1$ on nbhd of γ.

Cover γ by smaller discs $D_\varepsilon(a_n)$, $a_n \in \gamma$.

Get $\psi_n \in C_0^\infty(D_{2\varepsilon}(a_n))$, $\psi_n \equiv 1$ on $D_\varepsilon(a_n)$.

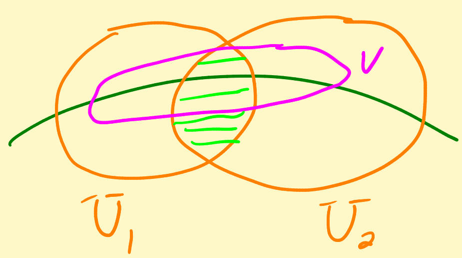
$\Psi = \sum_{n=1}^{N} \psi_n$ is C^∞ and >0 on a nbhd of γ.

Take $\varphi_n = X_\varepsilon \cdot \dfrac{\psi_n}{\Psi}$

shrink if necessary

Fact Solution to Cauchy prob: $\frac{\partial \varphi}{\partial \bar z} = v$, $\varphi|_\gamma = 0$

is locally unique:



agree on $U_1 \cap U_2$

get φ_1 get φ_2

Easy: $\frac{\partial}{\partial \bar{z}}(\varphi_1 - \varphi_2) = v - v \equiv 0$ there

Aha! $\varphi_1 - \varphi_2$ is analytic on $U_1 \cap U_2$ and vanishes on arc in open set (with limit pts).

So $\varphi_1 \equiv \varphi_2$ on $U_1 \cap U_2$.

Fact: Can solve Cauchy prob $\frac{\partial \varphi}{\partial \bar{z}} = v$, $\varphi|_\gamma = 0$ globally by piecing local sol^{n}s together using a C_0^∞ partition of unity.

Let $\{\psi_j\}_{j=1}^N$ be a C_0^∞ part of unity subordinate to cover by discs. Get local sol^{n}s φ_j.

$$\varphi = \sum_{j=1}^N \psi_j \varphi_j \quad \text{solves global prob.}$$

$$\frac{\partial \varphi}{\partial \bar{z}} = \underbrace{\sum_{j=1}^N \frac{\partial \psi_j}{\partial \bar{z}} \varphi_j}_{\text{sum is } \equiv 0 \text{ on a nbhd of } \gamma.} + \underbrace{\sum_{j=1}^N \psi_j \underbrace{\frac{\partial \varphi}{\partial \bar{z}}}_{v}}_{v \sum_1^N \psi_j}$$

ψ_j's add up to 1 on overlap!

agrees on overlap

$= v$ on a nbhd of γ