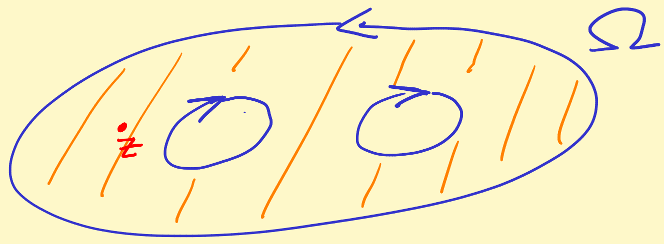
Lecture 12 The Cauchy transform



Ω bndd by C^∞-smooth curves.

u C^∞-smooth on $b\Omega$ (or merely continuous)

$$(\mathcal{C}u)(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z}\,dw$$

Today: $\mathcal{C} : C(b\Omega) \to \mathcal{O}(\Omega)$ ← analytic fcns on Ω.

Later: $\mathcal{C} : C^\infty(b\Omega) \to C^\infty(b\Omega)$

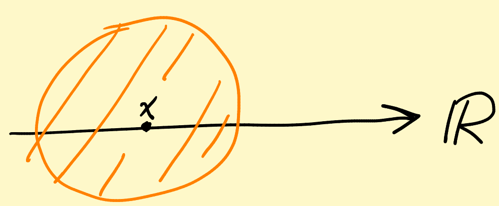
$\mathcal{C} : L^2(b\Omega) \to L^2(b\Omega)$

Application of Cauchy-Kovalevskaya thm to Cauchy transform

Suppose Ω has r.a. boundary curves and that u is a r.a. fcn on $b\Omega$.

Fact: u r.a. on $b\Omega \iff u = U\big|_{b\Omega}$ where U is a r.a. fcn on a nbhd of $b\Omega$.

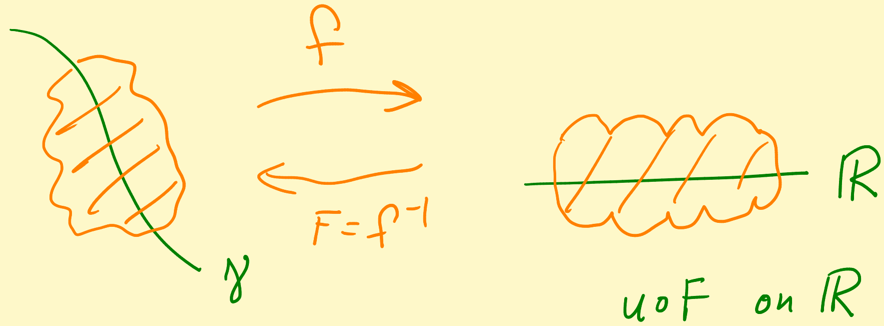
Sketch of pf Case



$u(x)$ r.a.

$$U(x,y) = u(x)$$

In general



$u \circ F$ on $\mathbb{R}$
extend it to $\mathcal{U}$ r.a.

u on γ

$U = \mathcal{U} \circ f$

<u>Thm</u> If Ω is bndd by r.a. curves and u is r.a. on $b\Omega$, then $\mathcal{C}u$ has boundary values that are r.a. on $b\Omega$.

<u>Pf</u> $u = U\big|_{b\Omega}$ where U is r.a. on a nbhd of $b\Omega$.

<u>Hmmm</u> Use $\chi \in C^\infty(\mathbb{C})$, $\chi \equiv 1$ on a smaller nbhd and χ supported near $b\Omega$, to extend U to $\mathbb{C}$.

$= u(w)$ on $b\Omega$ $\qquad$ r.a. near $b\Omega$

$$U(z) = \underbrace{\frac{1}{2\pi i}\int_{b\Omega} \frac{U(w)}{w-z}\,dw}_{\mathcal{C}u} + \frac{1}{2\pi i}\iint_{\Omega} \frac{\frac{\partial U}{\partial \bar{w}}(w)}{w-z}\,dw \wedge d\bar{w}$$

Aha! Use C-K thm to get φ with φ r.a. near $b\Omega$, $\varphi\big|_{b\Omega} = 0$ and $\frac{\partial \varphi}{\partial \bar{z}} = \frac{\partial U}{\partial \bar{z}}$ on a nbhd of $b\Omega$.
Extend φ C^∞ to $\mathbb{C}$.

Replace $\bar{U}$ by $\bar{U}-\varphi$ in improved CIF.

$$\underbrace{\bar{U}(z)-\varphi(z)}_{\substack{\text{Let } z\to b\Omega.\\ C^\infty \text{ up to bndry!}\\ \text{on } b\Omega \,=\, u.}} = \underbrace{\frac{1}{2\pi i}\int_{b\Omega}\frac{\overbrace{\bar{U}(w)-\varphi(w)}^{u(w)}}{w-z}\,dw}_{(\mathcal{C}u)(z)} + \underbrace{\frac{1}{2\pi i}\iint_{\Omega}\frac{\overbrace{\frac{\partial\bar{U}}{\partial\bar{w}}-\frac{\partial\varphi}{\partial\bar{w}}}^{\in C_0^\infty(\Omega)}}{w-z}\,dw\wedge d\bar{w}}_{\substack{\text{analytic in } z\\ \text{near } b\Omega \text{ and}\\ \text{it extends analytically}\\ \text{past bndry.}}}$$

$$\underset{z\in b\Omega}{(\mathcal{C}u)(z)} = u(z) - \underbrace{\frac{1}{2\pi i}\iint_{\Omega}\frac{\Psi(w)}{w-z}\,dw\wedge d\bar{w}}_{\substack{\text{analytic near } b\Omega,\\ \text{so r.a.}}}$$

where $\Psi\in C_0^\infty(\Omega)$.

<u>Plan</u>: Get a C^k version of Cauchy-Kovalevskaya thm.

Given v $\mathcal{C}^\infty$-smooth on a nbhd of $b\Omega$, get φ $\mathcal{C}^\infty$-smooth on a nbhd of $b\Omega$ such that

$\frac{\partial\varphi}{\partial\bar{z}} = v$ to high order k.

Let $\Omega=\{r<0\}$, $r\in\mathcal{C}^\infty(\mathbb{C})$, $\nabla r\neq 0$ on $b\Omega$.

Fact: If $\theta \in C^\infty(\mathbb{C})$, then θr^{k+1} vanishes to order k on $b\Omega$. In fact, θr^{k+1} extends from Ω to $\mathbb{C}$ as a C^k fcn with compact support in $\overline{\Omega}$ via extension by zero outside Ω.

How to cook up φ:

$$\varphi = \theta_0 r + \theta_1 r^2 + \theta_2 r^3 + \cdots + \theta_k r^{k+1}$$

Step 1 $\quad \Phi_0 = \theta_0 r$

want

$$\text{Want} \quad \frac{\partial}{\partial \bar z} \Phi_0 = \underbrace{\theta_{0\bar z}\, r}_{0 \text{ on } b\Omega} + \theta_0 r_{\bar z} \overset{\text{want}}{\cong} V$$

Big idea: Take

$$\theta_0 = \chi \frac{V}{r_{\bar z}}$$

χ: C^∞ fcn $\equiv 1$ near $b\Omega$, $\equiv 0$ outside a collar about $b\Omega$.

$$\frac{\partial}{\partial \bar z}\Phi_0 - V = \theta_{0\bar z}\, r + (\text{supported away from } b\Omega)$$

$$= \Psi_0 r \quad \text{where } \Psi_0 \text{ is } C^\infty.$$

First step : $\Phi_0 = \theta_0 r$ $\qquad \dfrac{\partial \Phi_0}{\partial \bar z} - v = \Psi_0 r$

Step 2 $\qquad \Phi_1 = \theta_0 r + \theta_1 r^2 = \Phi_0 + \theta_1 r^2$

$$\frac{\partial}{\partial \bar z} \Phi_1 = \frac{\partial \Phi_0}{\partial \bar z} + \theta_{1\bar z} r^2 + \theta_1 2r r_{\bar z}$$

$$\frac{\partial \Phi_1}{\partial \bar z} - V = \underbrace{\frac{\partial \Phi_0}{\partial \bar z} - V}_{\Psi_0 r} + \theta_{1\bar z} r^2 + \theta_1 2r r_{\bar z}$$

Aha! Take

$$\theta_1 = \chi \cdot \left(\frac{-\Psi_0}{2 r_{\bar z}} \right)$$

$$= \Psi_1 r^2 \quad \text{where } \Psi_1 \in C^\infty.$$

Repeat! $\quad$ Get $\Phi_k = \theta_0 r + \theta_1 r^2 + \cdots + \theta_k r^{k+1}$ $\leftarrow$ $=0$ on $b\Omega$

$$\frac{\partial \Phi_k}{\partial \bar z} - V = \Psi_k r^{k+1}$$ $\leftarrow$ vanishes to order k on $b\Omega$.

See page 7 of book for all details.