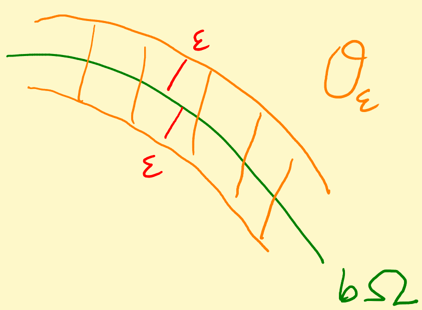
# Lecture 13 Boundary behavior of the Cauchy transform

<u>Thm</u>: $\Omega$ a domain with $C^\infty$-smooth boundary curves and $v \in C^\infty(\bar\Omega)$. Given $k \in \mathbb{N}$, there is a fcn $\varphi \in C^\infty(\bar\Omega)$ that vanishes on $b\Omega$ such that $\frac{\partial\varphi}{\partial\bar z} - v$ vanishes to order $k$ on $b\Omega$.

<u>Last time</u>: $\Omega = \{r < 0\}$, $b\Omega = \{r = 0\}$, $\nabla r \neq 0$ on $b\Omega$.

Note: $r_{\bar z} = \frac{1}{2}\left(\frac{\partial r}{\partial x} + i\frac{\partial r}{\partial y}\right)$, so $r_{\bar z} \neq 0$ on $b\Omega$ too.




$\chi \in C_0^\infty(\mathcal{O}_\varepsilon)$, $\chi \equiv 1$ on nbhd of $b\Omega$.

$r_{\bar z} \neq 0$ on $\mathcal{O}_\varepsilon$.

$\Phi_0 = \theta_0 r$.

$$\frac{\partial\Phi_0}{\partial\bar z} = \theta_{0\bar z} r + \theta_0 r_{\bar z} \overset{\text{want}}{\cong} v$$

$$\frac{\partial\Phi_0}{\partial\bar z} - v = \underbrace{\theta_{0\bar z} r}_{\text{good}} + \underbrace{\theta_0 r_{\bar z} - v}_{\text{Let } \theta_0 = \chi\cdot\frac{v}{r_{\bar z}}}$$

$\underbrace{\qquad\qquad}_{\text{this} \equiv 0 \text{ in nbhd of } b\Omega}$

$= \Psi_0 r$ where $\Psi_0 \in C^\infty(\bar\Omega)$.

Last time, showed how to get $\Phi_1 = \Phi_0 + \theta_1 r^2$ such that $\frac{\partial\Phi_1}{\partial\bar z} - v = \Psi_1 r^2$ where $\Psi_1 \in C^\infty(\bar\Omega)$.

Induction step  Suppose we have gotten

$\Phi_k$, $k=1,2,\dots,m-1$ such that

$$\Phi_k = \theta_0 r + \theta_1 r^2 + \cdots + \theta_k r^{k+1}$$ and

$$\frac{\partial \Phi_k}{\partial \bar z} - V = \Psi_k\, r^{k+1}$$ ← vanishes to order $k$ on $b\Omega$.

↑ $\Psi_k \in C^\infty(\bar\Omega)$ for each $k$.

Next: $\Phi_m = \Phi_{m-1} + \theta_m r^{m+1}$

$$\frac{\partial \Phi_m}{\partial \bar z} - V = \underbrace{\frac{\partial \Phi_{m-1}}{\partial \bar z} - V}_{\Psi_{m-1} r^m} + \underbrace{(\theta_m)_{\bar z}\, r^{m+1}}_{\text{good!}} + \underbrace{\theta_m (m+1) r^m r_{\bar z}}$$

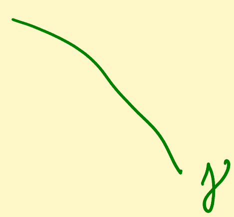
Kill! Make sum of these two terms vanish near $b\Omega$.

Aha! Take $\theta_m = \dfrac{-\Psi_{m-1}}{(m+1)\, r_{\bar z}} \cdot \chi$.

See $$\frac{\partial \Phi_m}{\partial \bar z} = \Psi_m\, r^{m+1}$$ where $\Psi_m \in C^\infty(\bar\Omega)$.

Remark  How to cook up $r$.

$\gamma : z(t) = x(t) + i y(t)$



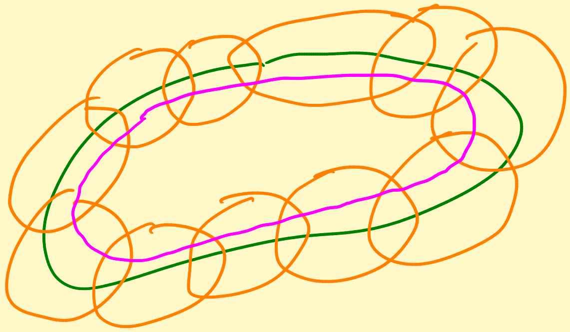

Use implicit fcn thm to get $t = X^{-1}(x)$ or $t = Y^{-1}(y)$.

Realize $\gamma$ as a graph of $\underbrace{y = f(x)}$ or $\underbrace{x = g(y)}$

$r(x,y) = f(x) - y$ $\qquad$ $r(x,y) = g(y) - x$

$r_j$

Get $r$ locally. Use $C^\infty$ partition of unity to get a global $r$.



$r_N \equiv -1$ on big compact set as shown

$r_N \in C_0^\infty(\Omega)$.

$\{\varphi_j\}_{j=1}^N$ $C_0^\infty$ partition of unity subordinate to open cover.

$$r = \sum_{j=1}^N \varphi_j r_j$$

Thm: Recall that if $v \in C_0^\infty(\mathbb{C})$, then

$$u(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{v(w)}{w - z} \, dw \wedge d\bar{w}$$

is $C^\infty$ on $\mathbb{C}$ and $\frac{\partial u}{\partial \bar{z}} = v$ on $\mathbb{C}$.

Proof showed that, if $v \in C_0^k(\mathbb{C})$, then $u$ is also a $C^k$ smooth sol$^n$ to $\frac{\partial u}{\partial \bar{z}} = v$.

**Thm:** If $\Omega$ is a $C^\infty$ smooth domain

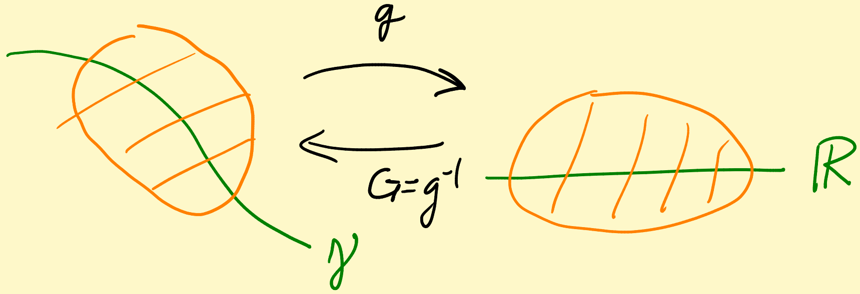
(a bounded domain bounded by finitely many non-intersecting $C^\infty$ Jordan curves)

and $u \in C^\infty(b\Omega)$, then $\mathcal{C}u$ is an analytic fcn on $\Omega$ that is $C^\infty$-smooth up to $b\Omega$.

**Pf:** Step 1: Extend $u$ to $\overline{\Omega}$ as a $C^\infty$ fcn:

$U \in C^\infty(\overline{\Omega})$ with $U\big|_{b\Omega} = u$.

How:




$u$ on $\gamma$

Get extension of $u$ to open set via $\mathcal{U} \circ g$.

$u \circ G$ is $C^\infty$ fcn of $x \in \mathbb{R}$

Extension $\mathcal{U}(x,y) = (u \circ G)(x)$

CIF

$$U(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{\overbrace{U(w)}^{=u(w) \text{ on } b\Omega}}{w-z}\,dw + \frac{1}{2\pi i}\iint_{\Omega} \frac{\frac{\partial U}{\partial \bar w}}{w-z}\,dw\wedge d\bar w$$

Aha! Given $k \in \mathbb{N}$, get $\varphi \in C^\infty(\overline{\Omega})$ with $\varphi\big|_{b\Omega} = 0$

with $\underbrace{\underbrace{\frac{\partial U}{\partial \bar{w}}}_{\text{think } v} - \frac{\partial \varphi}{\partial \bar{w}}}_{\Psi}$ vanishes to order $K$ on $b\Omega$

$\Psi = \Theta\, r^{K+1}$

CIF for $U - \varphi$ :

$$\underbrace{U(z) - \varphi(z)}_{C^\infty(\bar{\Omega})} = \underbrace{\frac{1}{2\pi i} \int_{b\Omega} \frac{\overbrace{U(w) - \varphi(w)}^{u(w)}}{w - z}\, dw}_{\mathcal{C}u} + \underbrace{\frac{1}{2\pi i} \iint_{\Omega} \frac{\overbrace{\frac{\partial U}{\partial \bar{w}} - \frac{\partial \varphi}{\partial \bar{w}}}^{\Psi}}{w - z}\, dw \wedge d\bar{w}}_{\text{Aha!}}$$

$\Psi$ extends to $\mathbb{C}$ as $C^K$ fcn with compact support in $\bar{\Omega}$ via extension by zero.

Know this is $C^K$ smoooth on $\mathbb{C}$ and solves a $\bar{\partial}$- problem.

Solve for $\mathcal{C}u$. See $\mathcal{C}u$ is $C^K(\bar{\Omega})$ for each $K$.

$$\mathcal{C} : C(b\Omega) \longrightarrow \mathcal{O}(\Omega)$$

$$\mathcal{C} : C^\infty(b\Omega) \longrightarrow C^\infty(\bar{\Omega}) \cap \mathcal{O}(\Omega)$$

Hmmm : $C^\infty(b\Omega) \longrightarrow C^\infty(b\Omega)$