Lecture 14 Applications of the Cauchy integral formula

$\boxed{\mathcal{C} : C^\infty(b\Omega) \to \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})}$

$u \in C^\infty(b\Omega)$. $\bar{U} \in C^\infty(\bar{\Omega})$ with $\bar{U}|_{b\Omega} = u$.

$\varphi \in C^\infty(\bar{\Omega})$, $\varphi|_{b\Omega} = 0$, $\underbrace{\frac{\partial \bar{U}}{\partial \bar{z}}}_{V} - \frac{\partial \varphi}{\partial \bar{z}}$ vanishes to order K on $b\Omega$.

$$\underbrace{\bar{U}(z) - \varphi(z)}_{C^\infty(\bar{\Omega})} = \underbrace{\frac{1}{2\pi i} \int_{b\Omega} \frac{\overbrace{\bar{U}(w) - \varphi(w)}^{u}}{w - z} dw}_{\mathcal{C}u} + \underbrace{\frac{1}{2\pi i} \iint_{\Omega} \frac{\overbrace{\frac{\partial \bar{U}}{\partial \bar{w}} - \frac{\partial \varphi}{\partial \bar{w}}}^{\Psi}}{w - z} dw \wedge d\bar{w}}$$

Ψ extends to $\mathbb{C}$ as C^K fcn with compact support in $\bar{\Omega}$ via extension by zero.

Know this is C^K smoooth on $\mathbb{C}$ and solves a $\bar{\partial}$-problem.

Interesting observations: 1) $$\frac{\partial}{\partial \bar{z}} \left(\underbrace{\frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\Psi(w)}{w - z} dw \wedge d\bar{w}}_{\text{analytic on } \mathbb{C} - \bar{\Omega}} \right) = \underbrace{\Psi(z)}_{\substack{=0 \\ \text{on } \mathbb{C} - \bar{\Omega}.}}$$

2) CIF for φ:

$$\varphi(z) = 0 + \frac{1}{2\pi i} \iint_{\Omega} \frac{\frac{\partial \varphi}{\partial \bar{w}}}{w - z} dw \wedge d\bar{w}$$

Hmmm. What happens if I let z slide outside of $\overline{\Omega}$.

Green's: $$\underbrace{\int_{b\Omega} \frac{\varphi(w)}{\underbrace{w-z}} \, dw}_{LHS = 0} = \iint_\Omega \frac{\partial}{\partial \bar{w}} \left[\frac{\varphi(w)}{w-z} \right] d\bar{w} \wedge dw$$

$\frac{1}{w-z}$ analytic in w on $\overline{\Omega}$ when $z \in \mathbb{C} - \overline{\Omega}$

$$= \iint_\Omega \frac{1}{w-z} \frac{\partial \varphi}{\partial \bar{w}} \, d\bar{w} \wedge dw$$

$$= -\iint_\Omega \frac{\frac{\partial \varphi}{\partial \bar{w}}}{w-z} \, dw \wedge d\bar{w} = 0$$

<u>Again:</u>

$$\underbrace{U(z) - \varphi(z)}_{C^\infty(\overline{\Omega})} = \underbrace{\frac{1}{2\pi i} \int_{b\Omega} \frac{\overbrace{U(w) - \varphi(w)}^{u}}{w-z} \, dw}_{\mathcal{C}u} + \underbrace{\frac{1}{2\pi i} \iint_\Omega \frac{\overbrace{\frac{\partial U}{\partial \bar{w}} - \frac{\partial \varphi}{\partial \bar{w}}}^{\Psi}}{w-z} \, dw \wedge d\bar{w}}$$

Think about z sliding up to $b\Omega$ and going out of Ω.

Last term: $$\frac{1}{2\pi i} \iint_\Omega \frac{\overbrace{\frac{\partial U}{\partial \bar{w}} - \frac{\partial \varphi}{\partial \bar{w}}}^{\Psi}}{w-z} \, dw \wedge d\bar{w}$$

$$= \begin{cases} H & \text{holo fcn on } \mathbb{C} - \overline{\Omega} \\ C^k(\overline{\Omega}) & \text{fcn inside } \Omega \end{cases}$$

Aha!

$$\boxed{H(z) = \frac{1}{2\pi i} \iint_{\Omega} \frac{\frac{\partial U}{\partial \bar{w}}}{w - z}\, dw \wedge d\bar{w}} \quad z \in \mathbb{C} - \bar{\Omega}$$

<u>Theorem</u> The $H(z)$ is independent of the choice of φ.

So $H(z)$ is C^∞-smooth up to $b\Omega$ from outside of Ω.

$$\underbrace{U(z) - \varphi(z)}_{C^\infty(\bar{\Omega})} = \underbrace{\frac{1}{2\pi i} \int_{b\Omega} \frac{\overbrace{U(w) - \varphi(w)}^{u}}{w - z}\, dw}_{\mathcal{C}u} + \underbrace{\frac{1}{2\pi i} \iint_{\Omega} \frac{\overbrace{\frac{\partial U}{\partial \bar{w}} - \frac{\partial \varphi}{\partial \bar{w}}}^{\Psi}}{w - z}\, dw \wedge d\bar{w}}$$

Let z slide to $b\Omega$: See

<u>Thm</u> Given $u \in C^\infty(b\Omega)$ on a smooth bounded domain Ω, there exist two analytic fcns:

1) $h \in \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})$, $h = \mathcal{C}u$.

2) H analytic on $\mathbb{C} - \bar{\Omega}$ and C^∞-smooth up to $b\Omega$ <u>and</u> $H(z) \to 0$ as $z \to \infty$.

such that $u = h + H$ on $b\Omega$.

Fun thing: On $D_1(0)$, get

$$u(z) = h(z) + H(\bar{z})$$

Hmmm, Schwarz fcn $S(z) = 1/z$

$\boxed{S(z) = \bar{z}}$ on $C_1(0)$.

$\bar{z} = \frac{1}{z}$ on $C_1(0)$

$$u(z) = \underbrace{h(z)}_{\substack{\text{analytic} \\ \text{in } D_1(0)}} + \underbrace{H(1/\bar{z})}_{\substack{\text{conjugate analytic} \\ \text{in } D_1(0) \\ \text{and } \to 0 \text{ as} \\ z \to 0.}} \quad \text{when } z \in C_1(0).$$

(0 is removable sing.)

$\frac{1}{\bar{z}} : D_1(0) \xrightarrow[\text{onto}]{1\text{-}1} \mathbb{C} - \overline{D_1(0)}$

Aha! Just solved D. Prob: $h(z) + H(1/\bar{z})$ is harmonic on $D_1(0)$ and has C^∞-smooth bndry values given by u!

Thm C^∞ regularity of D. Prob for unit disc.

Another fun thing $\quad u = h(z) + \underbrace{H(1/\bar{z})}_{\overline{h(z)}}$

$$u = h + \bar{h} \text{ on } C_1(0)$$

Hmmm. Assume u is real valued.

Whoa! h and $\bar{h}$ have same imag part on $C_1(0)$.

$\mathrm{Im}(h+\bar{h})$ is harmonic fcn with <u>zero</u> bndry values on $C_1(0)$, and so $\equiv 0$ on $D_1(0)$.

C-R eqns show: Two analytic fcns with same imag part differ by a const.

<u>Why</u> $\quad u_1 + iV = u_2 + iV$

$$\text{C-R eqns} \Longrightarrow \nabla u_1 \equiv \nabla u_2 \checkmark$$

So, $\quad u = h(z) + [h(z) + c]$

real const

$\bar{h} = h + c$

$=0$ at 0 $\quad$ so $c = -h(0)$

$$= (2\,\mathrm{Re}\, h) + c$$

$$c = -h(0) = -\underset{C_1(0)}{\mathrm{Ave}}\ h$$

$$= -\underset{C_1(0)}{\mathrm{Ave}}\ u$$

<u>Thm</u> Sol^n to D. Prob on $D_1(0)$ is

$$2\,\mathrm{Re}\left[\frac{1}{2\pi i}\int_{C_1(0)} \frac{u(w)}{w-z}\,dw\right] - \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta$$

Fun thing: Show this spits out Poisson formula!

<u>Thm for free</u>: Given $v \in C^\infty(\bar{\Omega})$, then

$$u(z) = \frac{1}{2\pi i} \iint_{\Omega} \frac{v(w)}{w-z}\, dw \wedge d\bar{w} \qquad (*)$$

is in $C^\infty(\bar{\Omega})$ and satisfies $\frac{\partial u}{\partial \bar{z}} = v$ in Ω.

<u>Pf</u>: Get φ with $\varphi|_{b\Omega} = 0$ and $v - \frac{\partial \varphi}{\partial \bar{z}}$ vanishes to order k on $b\Omega$.

Subtract CIF for φ from $(*)$.

Get $$u - \varphi = \underbrace{\frac{1}{2\pi i} \iint_{\Omega} \frac{\Psi(w)}{w-z}}_{\text{solves } \bar{\partial}\text{-prob } \frac{\partial u}{\partial \bar{z}} = \Psi.}\, dw \wedge d\bar{w}$$

Take $\frac{\partial}{\partial \bar{z}}$: $\frac{\partial}{\partial \bar{z}}(u - \varphi) = \frac{\partial}{\partial \bar{z}} \iint_{\mathbb{C}} = \Psi$

$$\frac{\partial u}{\partial \bar{z}} - \cancel{\frac{\partial \varphi}{\partial \bar{z}}} = v - \cancel{\frac{\partial \varphi}{\partial \bar{z}}}$$

cancel

$$\frac{\partial u}{\partial \bar{z}} = v \quad \checkmark$$