Lecture 15 Free theorems from $u = h + H$

$h \in \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})$. $h = \mathcal{C}u$
$H \in \mathcal{O}(\mathbb{C} - \bar{\Omega}) \cap C^\infty(\mathbb{C} - \Omega)$,
$H(z) \to 0$ as $z \to \infty$

Finish proof of

Thm for free: Given $v \in C^\infty(\bar{\Omega})$, then

$$u(z) = \frac{1}{2\pi i} \iint_\Omega \frac{v(w)}{w - z}\, dw \wedge d\bar{w} \qquad (*)$$

is in $C^\infty(\bar{\Omega})$ and satisfies $\frac{\partial u}{\partial \bar{z}} = v$ in Ω.

Pf: Get φ with $\varphi|_{b\Omega} = 0$ and $v - \frac{\partial \varphi}{\partial \bar{z}}$ vanishes to order k on $b\Omega$.

Subtract CIF for φ from $(*)$.

$\Psi = v - \frac{\partial \varphi}{\partial \bar{w}}$

Get
$$u - \varphi = \underbrace{\frac{1}{2\pi i} \iint_\Omega \frac{\Psi(w)}{w - z}\, dw \wedge d\bar{w}}_{\substack{\mathcal{U} \text{ solves } \bar{\partial}\text{-prob } \frac{\partial \mathcal{U}}{\partial \bar{z}} = \Psi. \\ \mathcal{U} \text{ is } C^k(\bar{\Omega}).}}$$

$u \in C^k(\bar{\Omega})$, $\forall k$. ✓

Take $\frac{\partial}{\partial \bar{z}}$: $\quad \frac{\partial}{\partial \bar{z}}(u - \varphi) = \frac{\partial}{\partial \bar{z}} \iint_{\mathbb{C}} = \Psi$

$$\frac{\partial u}{\partial \bar{z}} - \cancel{\frac{\partial \varphi}{\partial \bar{z}}} = \overbrace{v - \cancel{\frac{\partial \varphi}{\partial \bar{z}}}}^{\Psi}$$

cancel

$$\frac{\partial u}{\partial \bar{z}} = v \qquad \checkmark$$

Famous theorem The "jump" of the Cauchy transform of u as you go from z inside Ω to z outside Ω is u.

Last time: $\varphi|_{b\Omega} = 0$, z outside $\overline{\Omega}$

$$0 = \int_{b\Omega} \frac{\overbrace{\varphi(w)}^{\varphi=0}}{w-z}\,dw \overset{\text{Green's}}{=} \iint_\Omega \frac{\partial}{\partial \bar w}\left[\frac{\varphi(w)}{w-z}\right] d\bar w \wedge dw$$

$$= \iint_\Omega \frac{1}{w-z}\frac{\partial \varphi}{\partial \bar w}\, d\bar w \wedge dw$$

$$= -\iint_\Omega \frac{\frac{\partial \varphi}{\partial \bar w}}{w-z}\, dw \wedge d\bar w = 0$$

Repeat using U in place of φ; $U|_{b\Omega} = u$, z outside $\overline{\Omega}$

$$\underbrace{\int_{b\Omega} \frac{\overbrace{U(w)}^{=u(w)}}{w-z}\,dw}_{\text{want to see "jump"}} = \iint_\Omega \frac{\partial}{\partial \bar w}\left[\frac{U(w)}{w-z}\right] d\bar w \wedge dw$$

$$= \iint_\Omega \frac{1}{w-z}\frac{\partial U}{\partial \bar w}\, d\bar w \wedge dw$$

$$= -\iint_\Omega \frac{\frac{\partial U}{\partial \bar w}}{w-z}\, dw \wedge d\bar w$$

!

Conclude $$\frac{1}{2\pi i}\iint_\Omega \frac{\frac{\partial U}{\partial \bar w}}{w-z}\, dw \wedge d\bar w = -\frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z}\,dw$$

z outside!

One more time!

z inside

$$u(z) - \varphi(z) = (\mathcal{C}u)(z) + \frac{1}{2\pi i}\iint_{\Omega} \frac{\overbrace{\frac{\partial v}{\partial \bar w} - \frac{\partial \varphi}{\partial \bar w}}^{\Psi}}{w - z}\, dw \wedge d\bar w$$

z outside $\longrightarrow$ $-\frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z}\, dw = H(z)$

Theorem $u = h + H$ where $h = (\mathcal{C}u)(z)$, z inside Ω

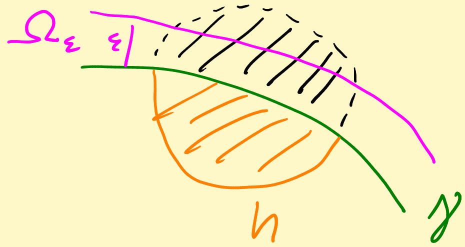
$$H = -\frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z}\, dw, \quad z \text{ outside } \Omega$$

This shows that u = jump of $\mathcal{C}u$ to $\mathcal{C}_{\text{outside}}\, u$.

Classic M-L argument: "If you can solve a problem in C^k for each k, you can do it in C^∞"

Thm: Ω bounded C^∞-domain, $v \in C^\infty(\bar\Omega)$, then there is a φ in $C^\infty(\bar\Omega)$ with $\varphi|_{b\Omega} = 0$ and $v - \frac{\partial \varphi}{\partial \bar z}$ vanishes to ∞ order on $b\Omega$.

Pf: $u = h + H$ Big idea: Let h_e be an extension of h past $b\Omega$ to all of $\mathbb{C}$ as a $C^\infty(\mathbb{C})$ fcn.



extend a little bit.
Then cut it off with
$\chi \in C_0^\infty(\Omega_\varepsilon)$, $\chi \equiv 1$ on nbhd of $\overline{\Omega}$.

<u>And</u>, let H_e be an extension of H to Ω as a C^∞ fcn on $\mathbb{C}$.

<u>Trivial observation</u>: Since $\frac{\partial}{\partial \bar z}$ of h_e is $\equiv 0$ inside Ω and $\frac{\partial}{\partial \bar z}$ of H_e is $\equiv 0$ outside Ω, it follows that $\frac{\partial h_e}{\partial \bar z}$ and $\frac{\partial H_e}{\partial \bar z}$ vanish to ∞ order on $b\Omega$.

Solution to "C^∞ - Cauchy problem".

<u>Step 1</u> Extend v to be $C_0^\infty(\mathbb{C})$.

Solve $\bar\partial$-problem $\frac{\partial u}{\partial \bar z} = v$ on $\mathbb{C}$.

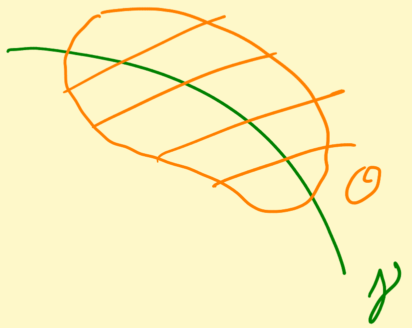
<u>Step 2</u> $\varphi = u - \underbrace{\left(h_e + H_e\right)}_{=0 \text{ on } b\Omega}$ solves problem!

$$\frac{\partial \varphi}{\partial \bar z} = \underbrace{\frac{\partial u}{\partial \bar z}}_{=v} + \text{something that vanishes to } \infty \text{ order on } b\Omega.$$

Thm: $u(z) = (Cu)(z) + \frac{1}{2\pi i} \iint_{\Omega} \frac{\Psi(w)}{w-z}\, dw \wedge d\bar{w}$ ← vanishes to ∞ order on $b\Omega$

when $z \in b\Omega$ where $\Psi \in C^\infty(\mathbb{C})$ with support on $\bar{\Omega}$.

<u>Remark</u>: Easy to adapt argument to get a local version =

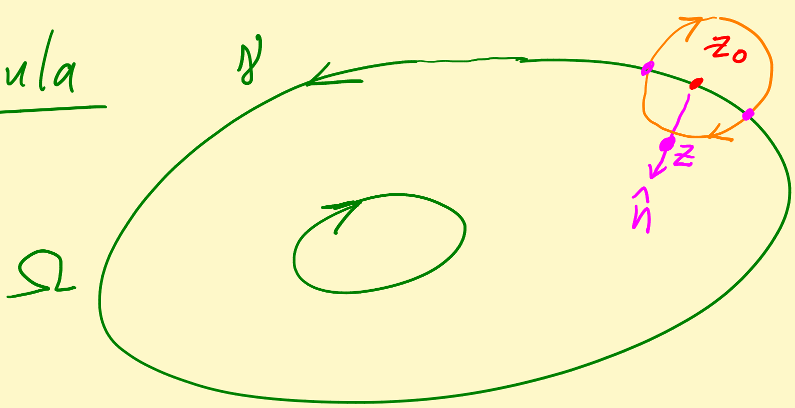


$V \in C^\infty(\mathcal{O})$

$\varphi|_\gamma = 0$, $\varphi \in C^\infty(\mathcal{O})$

$V - \frac{\partial \varphi}{\partial \bar{z}}$ vanishes to ∞ order on γ.

<u>Plemelj formula</u>

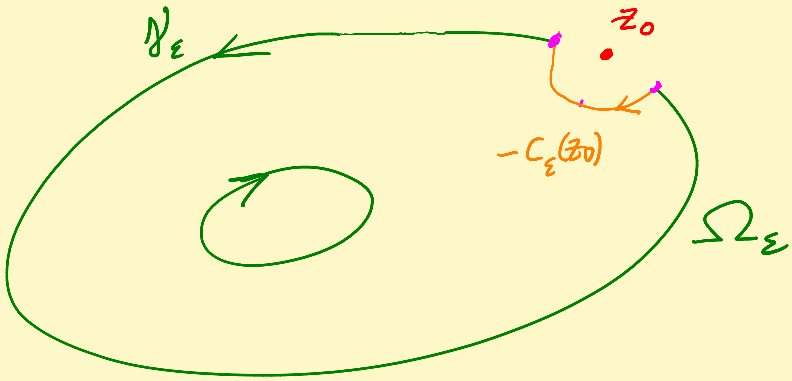


$b\Omega$ is C^2-smooth.

$u \in C^1$-smooth on $b\Omega$

$$\lim_{z \to z_0} (Cu)(z) = \frac{1}{2} u(z_0) + \text{P.V.} \frac{1}{2\pi i} \int_{b\Omega} \frac{u(w)}{w - z_0}\, dw$$

Get γ_ε, Ω_ε by punching out $D_\varepsilon(z_0)$



γ_ε is green part of boundary curve

$$PV \frac{1}{2\pi i} \int_\gamma \frac{u(w)}{w - z_0}\, dw = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{u(w)}{w - z_0}\, dw$$

Pf 1) True and easy if $u(z_0) = 0$.

2) True and easy if $u \equiv c$, a constant.

3) Easy trick $u(z) = \underset{\text{const}}{u(z_0)} + \Big[\underbrace{u(z) - u(z_0)}_{\text{vanishes at } z_0.}\Big]$

#2 $$\int_{\gamma_\varepsilon \cup -C_\varepsilon(z_0)} \frac{1}{z - z_0}\, dz = 0 \quad \text{by Cauchy's thm.}$$