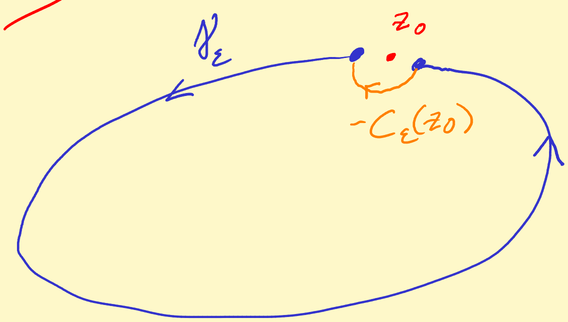
Lecture 16 Plemelj formula & pf of Runge's thm

C^∞ version: $(\mathcal{C}u)(z_0) = \frac{1}{2}u(z_0) + PV \frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z_0}\,dw$, $z_0 \in b\Omega$

Know bndry values smooth



Case 1: $u(z_0) = 0$. $(\mathcal{C}u)(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z}\,dw$ $\qquad \underset{\substack{\uparrow \\ z \text{ on normal}}}{\frac{u(w)}{w-z}} \xrightarrow{L^1} \underbrace{\frac{u(w)}{w-z_0}}_{L^1 \text{ fcn}}$

$$\xrightarrow[\substack{z\to z_0 \\ \text{along normal}}]{} \frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z_0}\,dw = PV$$

So $(\mathcal{C}u)(z_0) = \frac{1}{2}\underbrace{u(z_0)}_{0} + PV$ ✓

Case 2: $u \equiv c$. $0 = \int_{\gamma_\varepsilon \cup -C_\varepsilon(z_0)} \frac{c}{w-z_0}\,dw$ by Cauchy's thm

$$\int_{\gamma_\varepsilon} \frac{c}{w-z_0}\,dw = \int_{\underbrace{C_\varepsilon(z_0)}_{\text{"half circle"}}} \frac{c}{w-z_0}\,dw \xrightarrow[\varepsilon\to 0]{} \frac{1}{2}(2\pi i)\,\mathrm{Res}_{z_0}\frac{c}{z-z_0} = \pi i c$$

So $PV = \frac{1}{2}c$.

$$\underbrace{(\mathcal{C}u)(z_0)}_{\mathcal{C}u \equiv c} = \frac{1}{2}\underbrace{u(z_0)}_{c} + \underbrace{PV}_{\frac{1}{2}c} \quad ✓$$

General case follows from 1 & 2: $u(z) = \underbrace{u(z_0)}_{c} + \underbrace{[u(z) - u(z_0)]}_{=0 \text{ at } z_0}$

Hörmander's pf of Runge's : p.7 of his SCV book, chap. 1.

$C(K)$ = continuous complex valued fcns on K.
uniform topology.

Functional analysis (MA 546). Continuous linear fcn^{l} on $C(K)$ is given by $\varphi \longmapsto \int_K \varphi \, d\mu$ where μ is a complex valued Borel measure.

Two closed subspaces of $C(K)$

1) Closure of $\mathcal{O}(\Omega)\big|_K$ in $C(K)$.

2) Closure of fcns h analytic on any open set U with $K \subset U$.

Runge reduces to showing that every Borel measure μ $\perp$ #1 is $\perp$ #2 too. !!