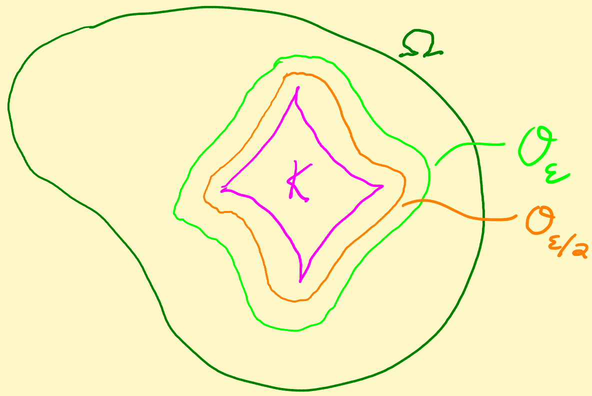
Lecture 17 The adjoint of the Cauchy transform

[The "formal" adjoint of the Cauchy transform as an operator on $C^\infty(b\Omega)$.]



Hörmander's Runge pf:

f analytic on $\mathcal{O}_\varepsilon$

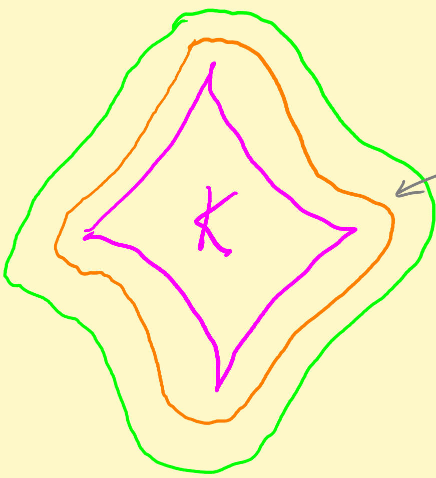
Step 1 Cook up $\chi \in C_0^\infty(\mathcal{O}_\varepsilon)$ with $\chi \equiv 1$ on $\mathcal{O}_{\varepsilon/2}$.

Step 2 Write CIF for χf which is also in $C_0^\infty(\mathcal{O}_\varepsilon)$ for a big disc containing $\overline{\mathcal{O}_\varepsilon}$.

$$(\chi f)(z) = \underbrace{(\text{Cauchy integral on big circle})}_{\equiv 0}$$

$$+ \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial}{\partial \bar w}(\chi f)}{w - z}\, dw \wedge d\bar w$$

$\equiv 0$ on $\mathcal{O}_{\varepsilon/2}$

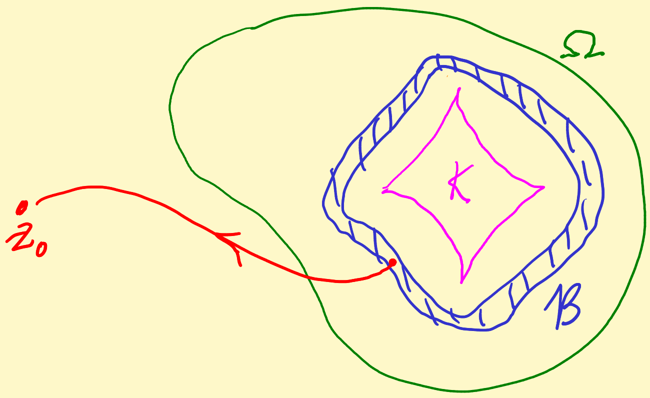


support of $\frac{\partial}{\partial \bar w}(\chi f)$ is contained in a narrow band $\mathcal{B}$ between $\mathcal{O}_{\varepsilon/2}$ and $\mathcal{O}_\varepsilon$.

Aha! When $z \in K$, get

$$f(z) = \frac{1}{2\pi i} \iint_{\mathcal{B}} \frac{[C_0^\infty(\mathcal{B}) \text{ fcn}]}{w - z}\, dw \wedge d\bar w$$

$\approx$ Riemann sum, showing that f can be uniformly approximated on K by rational fcns with poles in $\mathcal{B}$.



Classic Runge now allows poles to be moved out to z_0.

Fancy way: μ is a finite Borel measure on K.

$$M_1 = \text{Closure of } \mathcal{O}(\Omega)\big|_K \text{ in } C(K).$$

$$M_2 = \text{Closure of } \underbrace{A(K)}_{\text{fcns analytic on open set } \mathcal{O} \supset K \text{ restricted to } K.} \text{ in } C(K).$$

<u>Hahn-Banach</u>: If $f \in M_2 - M_1$, then $\exists$ cont lin fcn^l ℓ with

$$\ell(f) = 1 \text{ and } \ell\big|_{M_1} \equiv 0.$$

$$\ell(f) = \int_K f \, d\mu.$$ ← via Classic representation thm.

So, enough to show $\mu \perp M_1 \Rightarrow \mu \perp M_2$.

Hörmander: Express f as above using CIF.

$$\int_K f \, d\mu = \int_K \underbrace{\iint_{\mathcal{B}} \frac{[\overbrace{C_0^\infty \text{ fcn}}^{\psi}]}{w - z} \, dw \wedge d\bar{w}}_{f(z)} \, d\mu$$

$$\underset{\substack{\uparrow \\ \text{Fubini}}}{=} \iint_{\mathcal{B}} \left(\int_K \frac{\psi(w)}{w-z}\, d\mu(z) \right) dw \wedge d\bar{w}$$

Look analytic fcn $\displaystyle H(w) = \int_K \frac{1}{w-z}\, d\mu(z)$

analytic fcn of w on $\mathbb{C} - K$.

$$H^{(k)}(w) = c \int_K \underbrace{\frac{1}{(w-z)^{k+1}}}_{\text{analytic in a nbhd of } K.} d\mu(z)$$

Big idea: Let $w = z_0$$^{\in \mathbb{C}-\Omega}$. See Taylor series of H at z_0 has all zero coeff! So $H \equiv 0$ on connected component of $\Omega - K$, including $\mathcal{B}$. So $\ell(f) = 0$. ✓

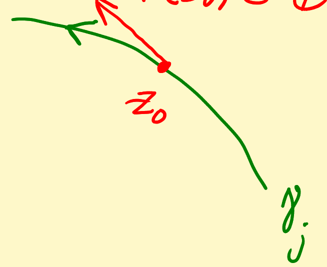
<u>Spaces</u> $L^2(b\Omega)$, $C^\infty(b\Omega)$

Boundary curves $\gamma_j : z_j(t)$, $0 \le t \le 1$.

Arc length $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = |z_j'(t)|\, dt$

$$dz = z_j'(t)\, dt$$

Complex unit tangent vector pointing in direction of the standard orientation $T(z_0) \in \mathbb{C}$ in $C'(0)$.



$$z_0 = z_j(t_0)$$

$$T(z_0) = \frac{z_j'(t_0)}{|z_j'(t_0)|}$$

Easy facts : $\quad dz = T\, ds$

A complex valued fcn u on $b\Omega$ is in $C^\infty(b\Omega)$ when $u(z_j(t))$ are C^∞ fcns on $[0,1]$ such that all derivatives agree at endpoints.

$L^2(b\Omega)$: $\quad \displaystyle\int_{b\Omega} |u|^2\, ds = \|u\|^2$

$$\langle u, v \rangle = \int_{b\Omega} u\, \bar{v}\, ds$$

<u>Fact</u> $C^\infty(b\Omega)$ is <u>dense</u> in $L^2(b\Omega)$.

<u>Know</u> : $\mathcal{C} : C^\infty(b\Omega) \to \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})$

$$\searrow C^\infty(b\Omega)$$

<u>Plan</u> $u, v \in C^\infty(b\Omega)$:

$$\langle \mathcal{C}u, v \rangle = \langle u, ? \rangle$$

$? =$ "formal adjoint of $\mathcal{C}$"

Result: $? = \mathcal{C}^* v = v - \overline{\mathcal{C}(\bar{v}\bar{T})} \cdot \overline{T}$

next time.