

Lecture 18 Adjoint of the Cauchy transform

1

Thm: $\langle Cu, v \rangle = \langle u, ? \rangle$

$? = v - \underbrace{C(\bar{v} \bar{T}) \cdot \bar{T}}_{C^*v}$ when $u, v \in C^\infty(b\Omega)$.

Key: $(Cu)(z) = u(z) - \underbrace{\frac{1}{2\pi i} \iint_{\Omega} \frac{\Psi(w)}{w-z} dw \wedge d\bar{w}}_{d(z)} \quad z \in b\Omega.$

where: 1) U is a C^∞ extension of u to $C^\infty(\bar{\Omega})$,

2) $\Psi = \frac{\partial U}{\partial \bar{z}} - \frac{\partial \varphi}{\partial \bar{z}} \in C^\infty(\bar{\Omega})$ and

Ψ vanishes to infinite order on $b\Omega$.

3) $\varphi \in C^\infty(\bar{\Omega})$ and $\varphi|_{b\Omega} \equiv 0$.

Pf: $\langle Cu, v \rangle = \langle u - d, v \rangle$

$= \langle u, v \rangle - \underbrace{\langle d, v \rangle}$

$\rightarrow = \int_{z \in b\Omega} \left(\frac{1}{2\pi i} \iint_{w \in \Omega} \underbrace{\frac{\Psi(w)}{w-z}}_{C^\infty(\bar{\Omega} \times b\Omega)} dw \wedge d\bar{w} \right) \overline{v(z)} ds_z$

Fubini $= \iint_{w \in \Omega} \Psi(w) \cdot \frac{1}{2\pi i} \int_{z \in b\Omega} \frac{\overline{v(z)}}{w-z} \underbrace{\overbrace{\bar{T}(z) T(z)}^{=1}}_{dz} ds_z dw \wedge d\bar{w}$

$$= \iint_{w \in \Omega} \Psi(w) H(w) dw \wedge d\bar{w}$$

where $\bar{H} = -\overline{C(\bar{v} \bar{T})}$ and $H \in \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})$.

$$\left(H(w) = \frac{1}{2\pi i} \int_{z \in \partial\Omega} \frac{\overline{v(z)} \overline{T(z)}}{z - w} dz \right) \quad \begin{matrix} w \in \Omega \\ z \in \partial\Omega \end{matrix}$$

$$= - \iint_{w \in \Omega} \underbrace{\left(\frac{2U}{2\bar{w}} - \frac{2\phi}{2\bar{w}} \right)}_{\frac{\partial}{\partial \bar{w}} [(U - \phi) H]} H(w) \underbrace{dw \wedge d\bar{w}}_{-d\bar{w} \wedge dw}$$

$\xrightarrow{\text{Green's}}$

$$\int_{w \in \partial\Omega} \underbrace{(U - \phi)}_{u=0} H dw = \int_{\partial\Omega} u H dw$$

$$= \int_{\partial\Omega} u \overline{H} \overline{T(w)} ds_w$$

$\underbrace{\hspace{1cm}}_{dw}$

$$= \langle u, \bar{H} \bar{T} \rangle \quad \text{Done!}$$

$$\langle \mathcal{C}u, v \rangle = \langle u, v \rangle - \langle u, \underbrace{\overline{\mathcal{C}(\bar{v}\bar{T})}}_{\bar{H}} \bar{T} \rangle$$

$$= \langle u, v - \overline{\mathcal{C}(\bar{v}\bar{T})} \bar{T} \rangle \quad \checkmark$$

Hardy space: $H^2(b\Omega)$ is the closure of

$$A^\infty(b\Omega) = \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega}) \Big|_{b\Omega} \quad \text{in } L^2(b\Omega).$$

Szegő projection: $H^2(b\Omega)$ is a closed subspace of $L^2(b\Omega)$,

so the Hilbert space projection $P: L^2(b\Omega) \rightarrow H^2(b\Omega)$
exists. Szegő' projection.

Lemma: If $h \in H^2(b\Omega)$, then $\bar{h}\bar{T} \perp H^2(b\Omega)$.

Pf: Suppose $h, H \in H^2(b\Omega)$.

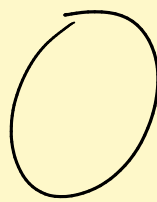
Know $\exists$ seq $h_j, H_j \in A^\infty(b\Omega)$ with

$$\begin{cases} h_j \rightarrow h & \text{in } L^2 \\ H_j \rightarrow H & \text{in } L^2 \end{cases}$$

$$\langle H, \bar{h}\bar{T} \rangle = \lim \langle H_j, \bar{h}_j\bar{T} \rangle$$

$$= \int_{\partial\Omega} H_j \overline{\bar{h}_j\bar{T}} \, ds$$

$$= \int_{\partial\Omega} H_j h_j \underbrace{\bar{T}}_{dz} \, ds$$

$\overset{=}{\uparrow}$ 
Cauchy's Theorem

Kerzman-Stein operator and identity

K-S operator: $C - C^* = A$

$$Au = Cu - [u - \overline{C(\bar{u}\bar{T})} \bar{T}]$$

$$= -u + \underbrace{h}_{Cu} + \underbrace{\bar{H}\bar{T}}_{H=C(\bar{u}\bar{T})}$$

So $\underbrace{u + Au}_{(I+A)u} = h + \underbrace{\bar{H}\bar{T}}_{P(\bar{H}\bar{T})=0} \leftarrow \text{orthogonal sum!}$

Take Szegő proj:

$$P(I+A) = Cu$$

Kerzman-Stein
identity

Big fact: A is a "smoothing operator" given by

$$(Au)(z) = \int_{w \in b\Omega} A(z, w) u(w) ds_w$$

where $A(z, w) = \frac{1}{2\pi i} \left(\frac{T(w)}{w - z} - \frac{\overline{T(z)}}{\bar{w} - \bar{z}} \right)$

Miracle! Singularities exactly cancel and

$$A \in C^\infty(b\Omega \times b\Omega)!$$