Lecture 18 Adjoint of the Cauchy transform

Thm: $\langle \mathcal{C}u, v\rangle = \langle u, ?\rangle$

$$? = \underbrace{v - \overline{\mathcal{C}(\bar{v}\,\bar{T})}\cdot \bar{T}}_{\mathcal{C}^* v} \quad \text{when } u, v \in C^\infty(b\Omega).$$

Key: $\displaystyle (\mathcal{C}u)(z) = u(z) - \underbrace{\frac{1}{2\pi i}\iint_\Omega \frac{\Psi(w)}{w-z}\, dw\wedge d\bar{w}}_{\mathcal{A}(z)} \qquad z\in b\Omega.$

where: 1) U is a C^∞ extension of u to $C^\infty(\bar{\Omega})$,

2) $\Psi = \dfrac{\partial U}{\partial \bar{z}} - \dfrac{\partial \varphi}{\partial \bar{z}} \in C^\infty(\bar{\Omega})$ and

Ψ vanishes to infinite order on $b\Omega$.

3) $\varphi \in C^\infty(\bar{\Omega})$ and $\varphi\big|_{b\Omega} \equiv 0$.

Pf: $\langle \mathcal{C}u, v\rangle = \langle u - \mathcal{A}, v\rangle$

$$= \langle u, v\rangle - \underbrace{\langle \mathcal{A}, v\rangle}$$

$$= \int_{z\in b\Omega}\left(\frac{1}{2\pi i}\iint_{w\in\Omega} \underbrace{\frac{\Psi(w)}{w-z}}_{C^\infty(\underset{w}{\bar{\Omega}}\times \underset{z}{b\Omega})} dw\wedge d\bar{w}\right)\overline{v(z)}\, ds_z$$

Fubini $$= \iint_{w\in\Omega} \Psi(w)\cdot \overline{\overline{\frac{1}{2\pi i}\int_{z\in b\Omega} \frac{\overline{v(z)}}{w-z}\, \overbrace{\bar{T}(z)T(z)}^{=1} \underbrace{ds_z}_{dz}}}\; dw\wedge d\bar{w}$$

$$= \iint_{w\in\Omega} \Psi(w) H(w)\, dw\wedge d\bar{w}$$

where $\bar{H} = -\overline{\mathcal{C}(\bar{v}\bar{T})}$ and $H \in \mathcal{O}(\Omega)\cap C^\infty(\bar{\Omega})$.

$$\left(H(w) = \frac{1}{2\pi i}\int_{z\in b\Omega} \frac{\overline{v(z)}\,\overline{T(z)}}{z-w}\, dz \right) \quad \begin{matrix} w\in\Omega \\ z\in b\Omega \end{matrix}$$

$$= -\iint_{w\in\Omega} \underbrace{\left(\frac{\partial \bar{v}}{\partial \bar{w}} - \frac{\partial \varphi}{\partial \bar{w}}\right) H(w)}_{\frac{\partial}{\partial \bar{w}}\left[(\bar{v}-\varphi)H\right]} \underbrace{dw\wedge d\bar{w}}_{-d\bar{w}\wedge dw}$$

$$\underset{\text{Green's}}{\overset{\uparrow}{=}} \int_{w\in b\Omega} \underbrace{(\bar{v}-\varphi)}_{u-0} H\, dw = \int_{b\Omega} u H\, dw$$

$$= \int_{b\Omega} u\, \overline{\overline{H}}\, \underbrace{T(w)\, ds_w}_{dw}$$

$$= \langle u, \bar{H}\bar{T} \rangle \qquad \text{Done!}$$

$$\langle \mathcal{C}u, v \rangle = \langle u, v \rangle - \langle u, \underbrace{\mathcal{C}(\bar{v}\bar{T})}_{\bar{H}}\, \bar{T} \rangle$$

$$= \langle u, v - \overline{\mathcal{C}(\bar{v}\bar{T})}\, \bar{T} \rangle \quad \checkmark$$

Hardy space: $H^2(b\Omega)$ is the closure of

$A^\infty(b\Omega) = \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})\Big|_{b\Omega}$ in $L^2(b\Omega)$.

Szegő projection: $H^2(b\Omega)$ is a closed subspace of $L^2(b\Omega)$, so the Hilbert space projection $\underbrace{P: L^2(b\Omega) \to H^2(b\Omega)}_{\text{Szegő projection.}}$ exists.

Lemma: If $h \in H^2(b\Omega)$, then $\overline{hT} \perp H^2(b\Omega)$.

Pf: Suppose $h, H \in H^2(b\Omega)$.

Know $\exists$ seq $h_j, H_j \in A^\infty(b\Omega)$ with

$$\begin{cases} h_j \to h & \text{in } L^2 \\ H_j \to H & \text{in } L^2 \end{cases}$$

$$<H, \overline{h}\overline{T}> = \lim <H_j, \overline{h_j}\overline{T}>$$

$$= \int_{b\Omega} H_j \overline{\overline{h_j}\overline{T}}\, ds$$

$$= \int_{b\Omega} H_j h_j \underbrace{T\,ds}_{dz} = 0$$

$\uparrow$ Cauchy's Thm

Kerzman-Stein operator and identity

K-S operator: $\mathcal{C} - \mathcal{C}^* = A$

$$Au = \mathcal{C}u - [u - \overline{\mathcal{C}(\bar{u}\overline{T})}\,\overline{T}]$$

$$= -u + \underset{\mathcal{C}u}{h} + \underset{H = \mathcal{C}(\bar{u}\overline{T})}{\overline{H}}\,\overline{T}$$

So $\underbrace{u + Au}_{(I+A)u} = h + \underbrace{\overline{HT}}_{P(\overline{H}\overline{T}) = 0}$ $\leftarrow$ orthogonal sum!

Take Szegő proj: $\boxed{P(I+A) = \mathcal{C}u}$ Kerzman-Stein identity

<u>Big fact</u>: A is a "smoothing operator" given by

$$(\mathcal{A}u)(z) = \int_{w \in b\Omega} A(z,w)\, u(w)\, ds_w$$

where $$A(z,w) = \underbrace{\frac{1}{2\pi i}\left(\frac{T(w)}{w-z} - \frac{\overline{T(z)}}{\bar{w}-\bar{z}} \right)}$$

Miracle! Singularities <u>exactly</u> cancel and

$A \in C^\infty(b\Omega \times b\Omega)$!