Lecture 19 The Kerzman-Stein kernel $A(z,w)$

Last time: $\mathcal{C}^* u = u - \overline{\mathcal{C}(\bar{u}\bar{T})}\,\bar{T}$ on $C^\infty(b\Omega)$:

$$\langle \mathcal{C}u, v\rangle = \langle u, \mathcal{C}^* v\rangle \text{ when } u,v\in C^\infty(b\Omega).$$

$A = \mathcal{C} - \mathcal{C}^*$ is the Kerzman-Stein operator.

Hardy space $H^2(b\Omega)$ is the closure of $A^\infty(b\Omega)$ in $L^2(b\Omega)$.

Orthogonal projection $P: L^2(b\Omega) \to H^2(b\Omega)$ is the Szegő projection.

Kerzman-Stein identity: $\boxed{P(I+A) = \mathcal{C}}$

Thm: Kerzman-Stein operator is a "smoothing operator":

$$(Au)(z) = \int_{w\in b\Omega} A(z,w)\, u(w)\, ds_w$$

where

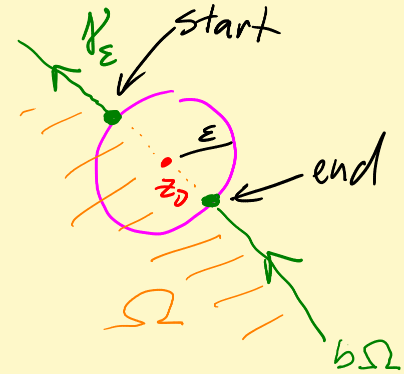
$$A(z,w) = \begin{cases} \dfrac{1}{2\pi i}\left(\dfrac{T(w)}{w-z} - \dfrac{\overline{T(z)}}{\bar{w}-\bar{z}}\right) & z\neq w \\ 0 & z=w \end{cases}$$

$z,w\in b\Omega$ is in $\mathcal{C}^\infty(b\Omega\times b\Omega)$.

Def^n The "Hilbert transform" $\mathcal{H}$ is defined

$$(\mathcal{H}u)(z_0) = PV \frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z_0}\,dw$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\gamma_\varepsilon}\frac{u(w)}{w-z_0}\,dw$$



Plemelj formula: $(\mathcal{C}u)(z_0) = \frac{1}{2}u(z_0) + (\mathcal{H}u)(z_0)$

Consequently, $\mathcal{C}: C^\infty(b\Omega)\circlearrowleft \Rightarrow$ so does $\mathcal{H}$.

Fact $Au = (\mathcal{C}u) - (\mathcal{C}^*u)$

$$= \mathcal{C}u - \left(u - \overline{\mathcal{C}(\bar{u}\bar{T})}\,\bar{T}\right)$$

$$\underset{\uparrow \text{Plemelj}}{=} \left[\tfrac{1}{2}u + \mathcal{H}u\right] - \left[u - \left(\tfrac{1}{2}u\right) - \overline{\mathcal{H}(\bar{u}\bar{T})}\,\bar{T}\right]$$

$$\frac{1}{2}\overline{(\bar{u}\bar{T})}\,\bar{T}$$

$$T\cdot\bar{T} = 1$$

Fact: $A u = \mathcal{H} u + \overline{\mathcal{H}(\bar{u}\,\bar{T})}\;\bar{T}$

Note $$(\mathcal{H} u)(z_0) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{u(w)}{w - z_0} \underbrace{T(w)\, ds_w}_{dw}$$

So,

$$(A u)(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \underbrace{\left(\frac{T(w)}{w - z} - \frac{\overline{T(z)}}{\bar{w} - \bar{z}} \right)}_{\text{no singularity at } z=w} u(w)\, ds_w$$

$$= \int_{w \in b\Omega} A(z,w)\, u(w)\, ds_w$$ [No P.V.!]

Proof that $A(z,w)$ is C^∞:

γ: $z(t)$: $0 \le t \le 1$ $\quad C^\infty$- parameterization.

Want: $A(z(s), z(t))$ is C^∞ in (s,t).

Take arc-length parameterization.

Recall, $T(z(t)) = \frac{z'(t)}{|z'(t)|} \leftarrow =1$ when $t=$arc-ℓ.

$$= z'(t).$$

<u>Lemma</u> If $X(s,t)$ is C^∞ on $(a,b)\times(c,d)$ and $X(s,s)=0$, then

$$X(s,t) = (s-t)\,Y(s,t) \quad \text{where } Y \text{ is } C^\infty$$

on $(a,b)\times(c,d)$.

<u>Pf</u>: $X(s,t) = X(s,t) - \overbrace{X(s,s)}^{0}$

$$= \int_s^t \frac{\partial X}{\partial v}(s,u)\,du \qquad \begin{cases} u = s+\tilde{\imath}(t-s) \\ du = (t-s)\,d\tilde{\imath} \end{cases}$$

$v =$ 2nd var

$$= \int_0^1 \frac{\partial X}{\partial v}\big(s, s+\tilde{\imath}(t-s)\big)\,(t-s)\,d\tilde{\imath}$$

$$= (t-s)\underbrace{\int_0^1 \frac{\partial X}{\partial v}\big(s, s+\tilde{\imath}(t-s)\big)\,d\tilde{\imath}}_{-Y(s,t)}$$

So $A(z(s), z(t)) = \boxed{\frac{1}{2\pi i}\left(\frac{z'(t)}{z(t)-z(s)} - \frac{\overline{z'(s)}}{\overline{z(t)} - \overline{z(s)}}\right)}$ $(*)$

Write

$$Q(s,t) = z(t) - z(s)$$

$$\underset{\text{Lemma}}{=} (t-s)\underbrace{X(s,t)}_{C^\infty \text{ smooth}}$$

Notice that $X(s,s) = \lim_{t\to s} \frac{z(t)-z(s)}{t-s} = \underbrace{z'(s)}_{\neq 0}$.

So $(*) = \frac{1}{2\pi i}\frac{1}{t-s}\left[\frac{z'(t)}{X(t,s)} - \frac{\overline{z'(s)}}{\overline{X(t,s)}}\right]$

denom's not zero when $s \approx t$.

On diagonal: $\left[\frac{z'(s)}{z'(s)} - \frac{\overline{z'(s)}}{\overline{z'(s)}}\right]$

$= 1 - 1 = 0$.

Use lemma again to cancel $\frac{1}{t-s}$ term!

Remark: Clear $A(z,w)$ is C^∞ smooth when $z \neq w$.
Only need to do analysis near $z=w$.

Have shown $A(z,w)$ is $C^\infty(b\Omega \times b\Omega)$.

Further similar analysis shows $A(z,z) = 0$.

<u>Fact</u> $A(z,w) = -\overline{A(w,z)}$, so

$A(z,w)$ is pure imaginary on diag.

<u>Theorem</u>: $P(I+A) = \mathcal{C}$ now shows that

$\mathcal{C}$ extends to $L^2(b\Omega)$ as a <u>bounded</u>

operator! And so $\mathcal{C}^*$ extends from

$C^\infty(b\Omega)$ to $L^2(b\Omega)$ and the extension

is the Hilbert space transform of $\mathcal{C}$.

Use $\mathcal{C}^*$ to denote the extension too.