Lecture 20 Consequences of Kerzman-Stein

$$A(z,w) = \begin{cases} \frac{1}{2\pi i}\left(\frac{T(w)}{w-z} - \frac{\overline{T(z)}}{\bar{w}-\bar{z}}\right), & z \neq w \\ 0 & z = w \end{cases}$$

Know $A \in C^\infty(b\Omega \times b\Omega)$.

$\mathcal{A} = \mathcal{C} - \mathcal{C}^*$ is smoothing operator: $L^2(b\Omega) \to C^\infty(b\Omega)$.

$$(\mathcal{A}u)(z) = \int_{w\in b\Omega} A(z,w)\, u(w)\, ds_w$$

and, of course, $\mathcal{A} : L^2(b\Omega) \circlearrowleft$ Bounded operator

Kerzman-Stein identity $\boxed{P(I+\mathcal{A}) = \mathcal{C}}$

Thm Since P, I, $\mathcal{A}$ are bounded operators on $L^2(b\Omega)$, $\mathcal{C}$ extends from $C^\infty(b\Omega)$ to $L^2(b\Omega)$ as a bounded operator.

Next: $\mathcal{C} : C^\infty(b\Omega) \circlearrowleft \implies$ same thing for P.

Hmmm. $$P(I+A) = \mathcal{C}$$
$$P + PA = \mathcal{C}$$
$$P = \mathcal{C} - \underbrace{PA}_{?}$$

Trick Adjoint of K-S identity.

Fact $P^* = P$ because $\perp$ projections are self adjoint.

$$[P(I+A)]^* = \mathcal{C}^*$$
$$(I+A)^* P^* \leftarrow A = \mathcal{C} - \mathcal{C}^*$$
$$A^* = \mathcal{C}^* - \mathcal{C} = -A$$

(A) $$\boxed{(I-A)P = \mathcal{C}^*} \leftarrow P - AP = \mathcal{C}^*$$

(B) $$P(I+A) = \mathcal{C} \leftarrow P + PA = \mathcal{C}$$

Hmmm: (B)−(A) : $$PA + AP = \underbrace{\mathcal{C} - \mathcal{C}^*}_{A}$$

Identity : $$\boxed{PA + AP = A}$$

Thm PA is a smoothing operator!

Pf: $PA = A - AP \leftarrow$ smoothing

Thm: $P : C^\infty(b\Omega) \circlearrowleft$

Next: What is $H^2(b\Omega)^\perp$ in $L^2(b\Omega)$?

Hint: Know $\bar{h}\bar{T} \perp H^2(b\Omega)$ when $h \in H^2(b\Omega)$.

smooth case: $\langle g, \bar{h}\bar{T} \rangle$ $\quad g, h \in A^\infty(b\Omega)$

$$= \int_{b\Omega} g\, \overline{\bar{h}\bar{T}}\, ds = \int_{b\Omega} g h\, \underbrace{T\, ds}_{dz}$$

$= 0$ by Cauchy's thm.

Used density of $A^\infty(b\Omega)$ in $H^2(b\Omega)$. ✓

Thm: $H^2(b\Omega)^\perp = \{ \bar{h}\bar{T} : h \in H^2(b\Omega)$

Pf: Suppose $v \perp H^2(b\Omega)$.

$$0 = \langle g, v \rangle$$

for $\forall\, g \in H^2(b\Omega)$. Big idea: Take $g = \mathcal{C}u$

where $u \in C^\infty(b\Omega)$.

$$0 = \langle \mathcal{C}u, v \rangle = \langle u, \mathcal{C}^* v \rangle$$

So $\mathcal{C}^* v \perp C^\infty(b\Omega)$, a <u>dense</u> subset of $L^2(b\Omega)$.

So $\mathcal{C}^* v \equiv 0$.

<u>Fact</u>: $\mathcal{C}^* v = v - \overline{\mathcal{C}(\bar{v}\,\bar{T})}\;\bar{T}$

extends to $v \in L^2(b\Omega)$

$$\mathcal{C}^* v = 0$$

$$v - \overline{\mathcal{C}(\bar{v}\,\bar{T})}\;\bar{T} = 0$$

$$v = \overline{\mathcal{C}(\bar{v}\,\bar{T})}\;\bar{T} = \bar{h}\,\bar{T}$$

where $h = \mathcal{C}(\bar{v}\,\bar{T})$ is in $H^2(b\Omega)$. ✓

Orthog decomposition thm

$u \in L^2(b\Omega)$:

$$(*) \quad u = \underbrace{h}_{Pu} + \underbrace{\bar{H}\bar{T}}_{P^\perp u}, \qquad h, H \in H^2(b\Omega)$$

Multiply $(*)$ by T

$$uT = hT + \bar{H}\underbrace{\bar{T}\cdot T}_{=1} = hT + \bar{H}$$

Take conjugate:

$$H + \bar{h}\bar{T} = \bar{u}\bar{T}$$

Aha! $\quad H = P(\bar{u}\bar{T})$.

Thm: $P^\perp u = \overline{P(\bar{u}\bar{T})}\,\bar{T}$

The Cauchy kernel

$$(\mathcal{C}u)(a) = \frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-a}\,dw$$

$$= \int_{b\Omega} u(w) \, \overline{\left[\frac{1}{2\pi i} \frac{1}{w-a} \cdot T(w) \right]} \, ds$$

$dw = T(w) ds_w$

$$= \langle u, C_a \rangle$$

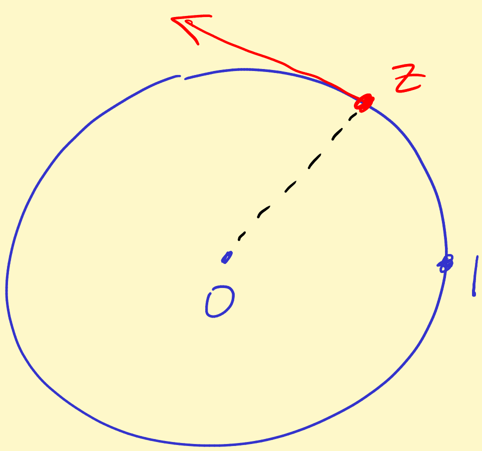
where $\quad C_a(w) = \dfrac{-1}{2\pi i} \dfrac{\overline{T(w)}}{\bar{w} - \bar{a}}$

is the "Cauchy kernel".

<u>Case $\Omega = D_1(0)$</u> $\qquad A(z,w) \equiv 0$!

K-S : $\qquad P = \mathcal{C}$

$T(z) = iz$



Hmmm: $\perp$ decomp: $\qquad u = h + \bar{H}\,\bar{T}$

$$= h + \bar{H}\,(\overline{iz})$$

$$= \underbrace{h - i\bar{z}\bar{H}}_{\text{solves D. Prob!}}$$

Thm: Solution to D. Prob in $D_1(0)$ sends $C^\infty(b\Omega) \to C^\infty(\bar{\Omega})$.

Interesting functional: Point evaluation of analytic fcns. $a \in \Omega$

$$h(a) = \text{Cauchy integral} = \langle h, C_a \rangle$$

Hmmm. $|h(a)| \le \|h\| \underbrace{\|C_a\|}_{< \infty}$

Bounded linear functional.

Riesz representation thm: So $\exists\, S_a \in H^2(b\Omega)$ so that $h(a) = \langle h, S_a \rangle$ ← the "Szegő kernel"

for all $h \in H^2(b\Omega)$.