Lecture 21 The Szegő Kernel

Calculation to show $A(z,z)=0$: <u>Arc length</u> parametrization:

$$A(z(t),z(s)) = \frac{1}{2\pi i}\left[\frac{z'(s)}{z(t)-z(s)} - \frac{\overline{z'(t)}}{\overline{z(t)}-\overline{z(s)}}\right]$$

$$= \frac{1}{2\pi i}\left[\frac{z'(s)\left[\overline{z(t)}-\overline{z(s)}\right] - \overline{z'(t)}\left[z(t)-z(s)\right]}{\left[z(t)-z(s)\right]\left[\overline{z(t)}-\overline{z(s)}\right]}\right]\cdot \frac{\frac{1}{(t-s)^2}}{\frac{1}{(t-s)^2}}$$

Let $t\to s$. Denom $\to z'(s)\overline{z'(s)}$ $\neq 0$. (In fact $=1$)

Use L'Hôpitals on Numerator <u>twice</u>:

$$\lim_{t\to s} \text{Num.} \overset{\frac{0}{0}}{\underset{LH}{=}} \lim_{t\to s} \frac{z'(s)\overline{z'(t)} - \overline{z''(t)}\left[z(t)-z(s)\right] - \overline{z'(t)}\,z'(t)}{2t-2s}$$

$$\overset{\frac{0}{0}}{\underset{LH}{=}} \lim_{t\to s} \frac{z'(s)\overline{z''(t)} - \overline{z'''(t)}\left[z(t)-z(s)\right] - \overline{z''(t)}z'(t) - \overline{z''(t)}z'(t) - \overline{z'(t)}z''(t)}{2}$$

$$= \frac{1}{2}\left[z'(s)\overline{z''(s)} - \overline{z''(s)}z'(s) - \overline{z''(s)}z'(s) - \overline{z'(s)}z''(s)\right]$$

$$= -\frac{1}{2}\left[\underbrace{\overline{z''(s)}\,z'(s) + \overline{z'(s)}\,z''(s)}\right]$$

$$\frac{d}{ds}\left[\underbrace{z'(s)\overline{z'(s)}}_{\equiv 1}\right] = 0 \; !$$

Last time : $h(a) = \langle h, C_a \rangle = \langle h, S_a \rangle$

(Riesz Rep Thm) $\exists\, S_a \in H^2(b\Omega)$

Fact: $S_a = P C_a$

Why: $h(a) = \langle h, C_a \rangle = \langle h, S_a \rangle$

$$\| \quad \langle h, PC_a \rangle$$

$\langle h, PC_a - S_a \rangle = 0$ for $h \in H^2(b\Omega)$.

both in $H^2(b\Omega)$

$PC_a - S_a \perp H^2(b\Omega) \Rightarrow PC_a - S_a = 0.$ ✓

Write $\underbrace{S(a,w)}_{\text{The Szegő kernel}} = \overline{S_a(w)}$

Fact $S(a,w)$ is the kernel for the Szegő projection P.

Why: Suppose $u \in L^2(b\Omega)$.

Then $Pu \in H^2(b\Omega)$.

Let $a \in \Omega$. Pu is analytic in Ω:

$$(Pu)(a) = \left(\mathcal{C}[Pu]\right)(a) = \langle Pu, C_a \rangle$$

$$= \langle Pu, \underbrace{PC_a}_{S_a} \rangle$$

$$= \langle u, S_a \rangle$$

$$= \int_{w \in b\Omega} u(w) \underbrace{\overline{S_a(w)}}_{S(a,w)} \, ds_w$$

$$(Pu)(a) = \int_{w \in b\Omega} S(a,w)\, u(w)\, ds_w \quad \checkmark$$

<u>Fact</u>: $S(a,b) = \overline{S(b,a)}$

Why: $\underbrace{\langle S_b, S_a \rangle}_{S_b(a)} = \underbrace{\overline{\langle S_a, S_b \rangle}}_{\overline{S_a(b)}}$

<u>Fact</u>: Consequence: $S(z,w)$ is analytic in z for fixed w, and antiholomorphic in w

for fixed z.

[See book for easy proof that $S(z,w) \in C^\infty(\Omega \times \Omega)$]

Garabedian kernel $L(z,a)$

Think about orthogonal decomp of C_a :

$$C_a = \underset{\substack{\uparrow \\ h = PC_a = S_a}}{h_a} + \overline{H_a T}, \quad h_a, H_a \in H^2(b\Omega)$$

$$-\frac{1}{2\pi i} \frac{\overline{T(w)}}{\bar{w} - \bar{a}} = S_a(w) + \overline{H_a(w)}\ \overline{T(w)}$$

Note: $a \in \Omega$, but $w \in b\Omega$.

Solve for S_a :

$$S_a(w) = -\frac{1}{i} \overline{\left[\frac{1}{2\pi} \frac{1}{w - a} + i H_a(w) \right]}\ \overline{T(w)}$$

$$= -\frac{1}{i} \overline{\left[\underbrace{\frac{1}{2\pi} \frac{1}{w - a} - i H_a(w)}_{L_a(w)} \right]}\ \overline{T(w)}$$

$$= -\frac{1}{i}\,\overline{L_a(w)}\;\overline{T(w)}$$

<u>Def^n</u>: $L_a(w) = \frac{1}{2\pi}\cdot\frac{1}{w-a} + h(w)$

as above, where $h \in H^2(b\Omega)$.

$$\boxed{\text{Write } \; L(z,a) = L_a(z)}$$

Garabedian Kernel

<u>Facts</u>: 1) $L(z,w)$ is analytic in z <u>and</u> w

on $\Omega \times \Omega - \{(w,w) : w \in \Omega\}$.

2) $L(z,w) = -L(w,z)$

3) $L(z,a)$ has a <u>simple pole</u> at $z=a$

with residue $\frac{1}{2\pi}$.

4) $$\boxed{\overline{S_a(w)} = \frac{1}{i}\,L_a(w)\,T(w)}$$

$a \in \Omega$
$w \in b\Omega$

Calculation

$$h(a) = \langle h, S_a \rangle$$

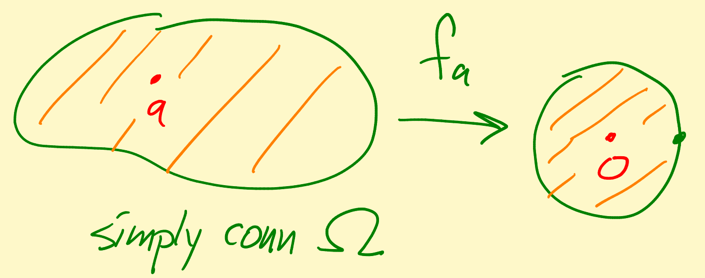
$$= \int_{b\Omega} h(w)\, \overline{S_a(w)}\, ds_w$$

$$= \int_{b\Omega} h(w) \left[\frac{1}{i} L_a(w) \underbrace{T(w) \right] ds_w}_{dw}$$

$$\underset{\#4}{=} \frac{1}{i} \int_{w \in b\Omega} h(w)\, L_a(w)\, dw$$

$$= \frac{1}{i} \left[2\pi i \ \mathrm{Res}_a \left(h L_a \right) \right] = \frac{1}{i} (2\pi i) \left[\frac{h(a)}{2\pi} \right] = h(a)$$

Mind blowing fact :



Riemann map f_a : $\Omega \xrightarrow[\text{onto}]{1-1} D_1(0)$ analytic

and $f_a'(a)$ is real and positie.

$(f_a'(a) > 0)$.

Fact

$$f_a(z) = \frac{S_a(z)}{L_a(z)} = \frac{S(z,a)}{L(z,a)}$$