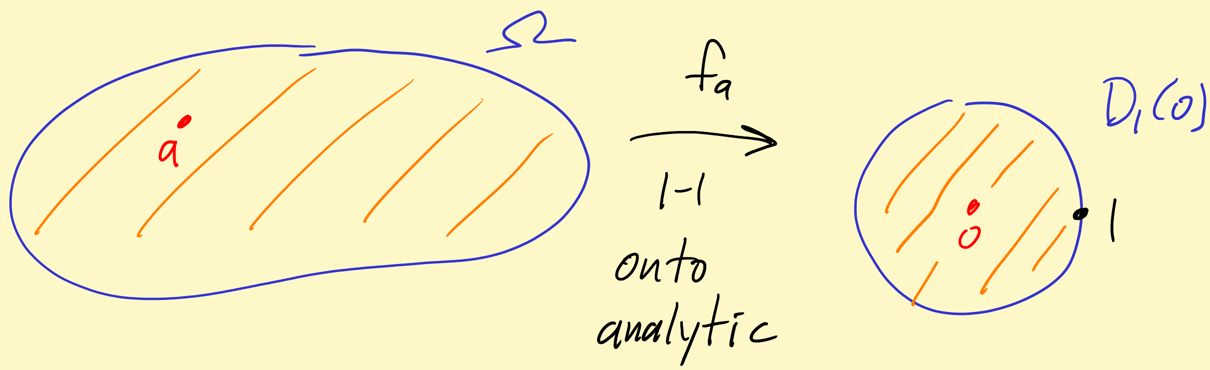
Lecture 22 The Riemann map & the Szegő kernel



Ω bounded by a C^∞-smooth Jordan curve, hence simply connected. $a \in \Omega$. $\begin{cases} f_a(a) = 0 \\ f_a'(a) \in \mathbb{R}^+ \end{cases}$

Thm: $f_a(z) = \dfrac{S(z,a)}{L(z,a)}$

Carathéodory's Thm: f_a is continuous up to $b\Omega$ (and $f : b\Omega \xrightarrow[\text{onto}]{1\text{-}1} C_1(0)$ and preserves counterclockwise sense). [See Stein & Sh. for a pf]

Easy facts: Recall $S_a(z) = S(z,a)$

1) $S_a \not\equiv 0$.

pf: $h \in H^2(b\Omega)$: $\quad h(a) = \langle h, S_a \rangle$

Let $h \equiv 1$. $\quad 1 = \langle 1, S_a \rangle$

So S_a not $\equiv 0$.

2) $S(a,a) = S_a(a) > 0$

Why: Let $h = S_a$. $\quad h(a) = \langle h, S_a \rangle$

$S_a(a) = \langle S_a, S_a \rangle$

So $S(a,a) = \|S_a\|^2 > 0$. ✓

Recall identity: $\boxed{\overline{S_a} = \frac{1}{i} L_a T}$

Facts: Last time $\left\{ \right.$ $S_a = P C_a$ is $C^\infty(\overline{\Omega})$. Consequently, L_a extends to C^∞-smoothly to $b\Omega$. It has a simple pole at $z=a$ with residue $\frac{1}{2\pi}$.

Note: $L(z,a) - \frac{1}{2\pi} \cdot \frac{1}{(z-a)}$ is in $H^2(b\Omega)$.

Lemma If $\sigma(z)$ and $\lambda(z)$ are such that

1) σ and $\lambda(z) - \frac{(1/2\pi)}{z-a}$ are in $H^2(b\Omega)$, and

2) $\overline{\sigma(z)} = \frac{1}{i} \lambda(z) T(z)$ on $b\Omega$,

then $\sigma = S_a$ and $\lambda = L_a$.

<u>Pf</u>: Hmmm. $\overline{S_a} = \frac{1}{i} L_a T$

$$\overline{\sigma} = \frac{1}{i} \lambda T$$

Subtract! $\overline{(S_a - \sigma)} = \underbrace{\frac{1}{i}(L_a - \lambda)}_{\in H^2(b\Omega)} T$

Let $h = S_a - \sigma$ and $H = \frac{1}{i}(L_a - \lambda)$.

Both in $H^2(b\Omega)$ and, taking conjugate:

$$\underset{\in H^2(b\Omega)}{h} = \underbrace{\overline{H}\,\overline{T}}_{\perp H^2(b\Omega)}$$

Conclude that $h \equiv 0$, $H \equiv 0$. ✓

<u>Proof that $f_a = \frac{S_a}{L_a}$</u>: Note that $|f_a(z)| = 1$

when $z \in b\Omega$. So $f_a(z)\overline{f_a(z)} = 1$

when $z \in b\Omega$.

$$f_a(z) = \frac{1}{\overline{f_a(z)}} \quad \text{on } b\Omega.$$

Hmmm.

$$\overline{S_a(z)} = \frac{1}{i} L_a(z) T(z)$$

Multiply:

$$\frac{\overline{S_a(z)}}{\overline{f_a(z)}} = \frac{1}{i} f_a(z) L_a(z) T(z)$$

Multiply by $\overline{T}$ and take conjugate

$$\frac{S_a}{f_a} T = -\frac{1}{i} \overline{f_a L_a}$$

Hmmm. Since $f_a(a) = 0$ and $f_a'(a) \neq 0$,

simple zero of f_a at a cancels simple pole of L_a at a. And, since $S_a(a) > 0$,

$\frac{S_a(z)}{f_a(z)}$ has a simple pole at $z = a$.

Choose $c \neq 0$ so that $\operatorname{Res}_a c \frac{S_a}{f_a} = \frac{1}{2\pi}$

$$c \frac{S(a,a)}{f_a'(a)} = \frac{1}{2\pi}$$

$$\boxed{c = \frac{f_a'(a)}{2\pi\, S(a,a)}} \leftarrow \in \mathbb{R}^+$$

$$\underbrace{\frac{1}{i}\left(c\,\frac{S_a}{f_a}\right)}_{\lambda} T = \underbrace{\overline{\left(f_a L_a\right)}}_{\bar{\sigma}}$$

Lemma $\Longrightarrow$ $\quad c\,\dfrac{S_a}{f_a} = L_a$

So $\quad f_a = c\,\dfrac{S_a}{L_a}$

Hmmm. What is c? Know $c = \dfrac{f_a'(a)}{2\pi\, S(a,a)} > 0$.

Pick $z \in b\Omega$ where $L(z,a) \neq 0$. $\quad f_a(z) = c\,\dfrac{S(z,a)}{L(z,a)}$

$$\overline{S(z,a)} = \frac{1}{i} L(z,a)\, T(z)$$

So $\quad |S(z,a)| = |L(z,a)| \neq 0$.

Since $|f_a(z)| = 1$, we see $|c| = 1$.

$c \in \mathbb{R}^+$. So $c = 1$. ✓

Thm: $\underbrace{f_a'(a)}_{\uparrow} = 2\pi\, S(a,a)$

extremal! among $h : \Omega \xrightarrow{\text{analytic}} D_1(0)$

Recap of some facts

$A = \mathcal{C} - \mathcal{C}^*$ is a smoothing operator.

Think: $\mathcal{C}$ is "almost" equal to $\mathcal{C}^*$.

$\mathcal{C}$ is "almost" self-adjoint.

$$\text{K-S:} \quad \begin{cases} P(I+A) = \mathcal{C} & (1) \\ PA + AP = A & (2) \end{cases}$$

$$(1): \qquad \mathcal{C} - P = PA \underset{(2)}{=} A - AP$$

$$\mathcal{C} - P = \boxed{A(I-P)}$$

$\leftarrow$ $L^2(b\Omega) \to C^\infty(b\Omega)$ is a smoothing operator

Think: Szegő proj and Cauchy transf are "almost" the same.

Unit disc: $A(z,w) \equiv 0$.

$$S(z,w) = \frac{1}{2\pi(1-z\bar{w})}$$

$$L(z,w) = \frac{1}{2\pi(z-w)}$$

$$\frac{S(z,a)}{L(z,a)} = \frac{z-a}{1-z\bar{a}} \quad \checkmark$$

$$S(z,w) = \frac{1}{2\pi}\sum_{n=0}^{\infty} \underbrace{(z^n \bar{w}^n)}$$

$\varphi_n(z) = \frac{z^n}{\sqrt{2\pi}}$

is an orthonormal basis for $H^2(C_1(0))$.