Lecture 23 Solving the Dirichlet problem using the Szegő projection P.

$$\boxed{\overline{S_a} = \frac{1}{i} L_a T}$$

Ω C^∞-smooth bounded by a C^∞-smooth Jordan curve. So Ω is simply connected.

Know $S_a = P C_a$ is in $A^\infty(\Omega) = \mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega})$.

Also $L(z,a) = \dfrac{1}{2\pi(z-a)} + H_a(z)$ where

$H_a \in A^\infty(\Omega)$. So $L_a(z)$ extends C^∞-smoothly to the boundary.

Important facts: 1) $S(z,a) \neq 0$ for $z \in \overline{\Omega}$ and $a \in \Omega$.

2) $L(z,a)$ has a simple pole at $z=a$ and

$L(z,a) \neq 0$ when $z \in \overline{\Omega} - \{a\}$ and $a \in \Omega$.

Proof in book. Technical, but not hard.

Game: Play with $\perp$ decomposition to get a harmonic fcn $h + \overline{H}$ that solves a D. Prob.

Trick: $\qquad \overline{S_a} = \frac{1}{i} L_a T$

Take conjugate: $\qquad S_a = -\frac{1}{i} \overline{L_a}\, \overline{T}$

Solve for $\overline{T} = -i \dfrac{S_a}{\overline{L_a}}$ on $b\Omega$.

Hmmm: $\qquad u = h + \overline{H}\,\overline{T}$ is $\perp$ decomp on $b\Omega$

$$u = h + \overline{H}\left(-i \frac{S_a}{\overline{L_a}}\right) \quad \text{on } b\Omega$$

Hmmm. Divide by S_a:

$$\frac{u}{S_a} = \underbrace{\frac{h}{S_a} - i\,\frac{\overline{H}}{\overline{L_a}}}_{\text{extends to } \Omega \text{ as a harm fcn!}} \quad \text{on } b\Omega$$

$\frac{h}{S_a} \in A^\infty$

$\frac{\overline{H}}{\overline{L_a}}$ also in $\overline{A^\infty}$ because pole of L_a at a yields a zero of $\frac{H}{L_a}$ at a.

extends to Ω as a harm fcn!

Get harmonic extension of $\frac{u}{S_a}$.

Aha! Start with $S_a u$ instead of u!

Get $S_a u = h + \overline{H}\,\overline{T}$

$h = P(S_a u)$. Similar for $\overline{H}$.

Play game: $u = \underbrace{\frac{h}{S_a} - i\frac{\overline{H}}{\overline{L}_a}}$

extends to Ω and solves D. prob with bndry values u.

<u>Thm</u>: In a simply connected C^∞-smooth bounded domain. Given $u \in C^\infty(b\Omega)$, the sol^n to D. Prob with bndry values u is

$$h + \overline{H}$$

where $h = \frac{P(S_a u)}{S_a}$ and $H = \frac{P(L_a \bar{u})}{L_a}$.

Consequently, the (unique) sol^n to D. prob is in $C^\infty(\overline{\Omega})$.

Another D. prob: Given $v \in C^\infty(\overline{\Omega})$, find u with $\Delta u = v$ on Ω with $u|_{b\Omega} \equiv 0$.

Important operator: Green's operator:

$$Gv = u$$

where $\Delta u = v$ and $u|_{b\Omega} \equiv 0$.

Thm: $G : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$.

Pf: Step 1: Extend v to $\mathbb{C}$ as a fcn in $C_0^\infty(\mathbb{C})$. Call extension v too.

Recall $\Delta = 4 \dfrac{\partial}{\partial z} \dfrac{\partial}{\partial \bar{z}}$

Given v, solve

1) Solve $\dfrac{\partial \mu}{\partial z} = \dfrac{1}{4} v$

Want

$\Delta u = v$

$4 \dfrac{\partial}{\partial z}\underbrace{\left(\dfrac{\partial u}{\partial \bar{z}}\right)}_{\mu} = v$

Know:

$$u(z) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{\frac{1}{4} v(w)}{w - z} \, dw \wedge d\bar{w}$$

Solves $\dfrac{\partial u}{\partial \bar{z}} = \dfrac{1}{4} v$

Fact $\overline{\dfrac{\partial u}{\partial \bar{z}}} = \dfrac{\partial \bar{u}}{\partial z}$.

Get sol^n $\displaystyle u(z) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{\frac{1}{4} v(w)}{\bar{w} - \bar{z}} \, dw \wedge d\bar{w}$

to $\dfrac{\partial u}{\partial z} = \dfrac{1}{4} v$.

Know $u \in C^\infty(\mathbb{C})$. Use a C_0^∞ fcn χ that is $\equiv 1$ on an open set containing $\overline{\Omega}$.

Replace u by χu.

Next, solve $\dfrac{\partial u}{\partial \bar{z}} = u$.

Get sol^n to $\Delta u = v$ on Ω with $u \in C^\infty(\overline{\Omega})$.

Finally, solve D. prob using Szegő projection for $\tilde{u}$ = boundary values of $\mathcal{U}$. Call it φ.

Sol^n we want is $\quad u = \mathcal{U} - \varphi$.

$$u\big|_{b\Omega} \equiv 0 \quad \checkmark$$

$$\Delta u = \underbrace{\Delta \mathcal{U}}_{v} - \underbrace{\Delta \varphi}_{0} \quad \checkmark$$

What is the relationship between $\mathcal{C}u$ inside Ω and its boundary values?

$H^2(b\Omega)$ is the closure of $A^\infty(b\Omega)$ in $L^2(b\Omega)$.

Given $u \in L^2(b\Omega)$: we took $u_j \in C^\infty(b\Omega)$ with $u_j \to u$ in $L^2(b\Omega)$.

K-S shows that $\mathcal{C}u_j \xrightarrow{L^2(b\Omega)} v \in L^2(b\Omega)$.

Define $v = \mathcal{C}u$

$$h(z) = \frac{1}{2\pi i} \int_{b\Omega} \frac{u(w)}{w-z}\, dw$$

is analtic inside Ω.

<u>Fact</u> h has L^2 boundary values given by $\mathcal{C}u$.

<u>Pf</u>: Book, Chap 6.