Lecture 24 Hardy space history (Chap. 6)

Remarks Showed Riemann map $f_a(z) = \dfrac{S(z,a)}{L(z,a)}$.

(Actually showed: $L_a f_a = S_a$. So zeroes of L_a in Ω would also have to be zeroes of S_a of same order, so they cancel in formula $f_a = S_a / L_a$. What about zeroes on bndry? Tricky! See Chapter 9 for proof of fact:

Thm: Suppose Ω is a bounded C^∞-smooth domain, finitely many bndry curves. Then $L(z,a)$ has no zeroes on $z \in \bar{\Omega} - \{a\}$, $a \in \Omega$.
If Ω is simply connected too, then $S(z,a)$ also has no zeroes on $\bar{\Omega} \times \Omega$.

Cor: Ω smooth, simp conn, bounded. Riemann map is in $C^\infty(\bar{\Omega})$.

Remark Sol^n to D. Prob: Harmonic extension of

$u \in C^\infty(b\Omega)$ is $h + \overline{H}$ where $h = \dfrac{P(S_a u)}{S_a}$

and $H = \dfrac{P(L_a \bar{u})}{L_a}$ are in $\mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega})$.

Should have started by saying: Pick a point $a \in \Omega$ and fix it.

History Started with $H^2(D_1(0))$. $\dfrac{z^n}{\sqrt{2\pi}}$ is an orthogonal basis. Orthogonal expansion $\sum_{n=0}^{\infty} c_n z^n$ is just a Taylor series.

$$\|h\|^2 = \frac{1}{2\pi} \sum_{n=0}^{\infty} |c_n|^2 \qquad H^2 \approx \ell^2$$

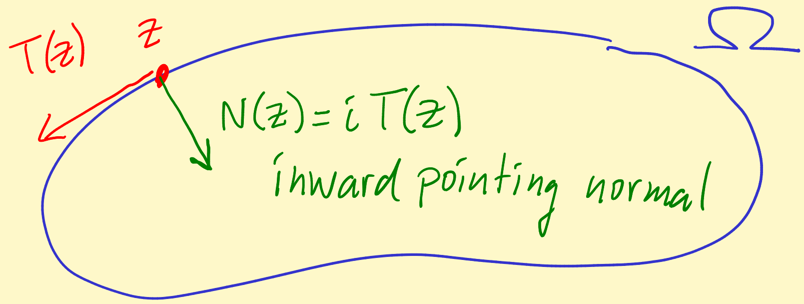
Hardy: $h \in H^2(D_1(0))$. Look at

$h_r(z) = h(rz)$, $0 < r < 1$. Showed that

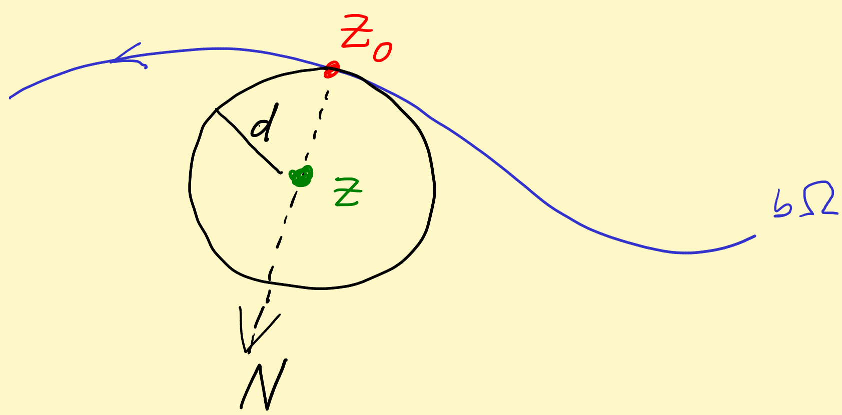
$$h_r(e^{i\theta}) = h(re^{i\theta}) \xrightarrow[L^2[0,2\pi]]{} h(e^{i\theta}) \text{ as } r \nearrow 1.$$

Hardy space fcns have L^2 bndry values.

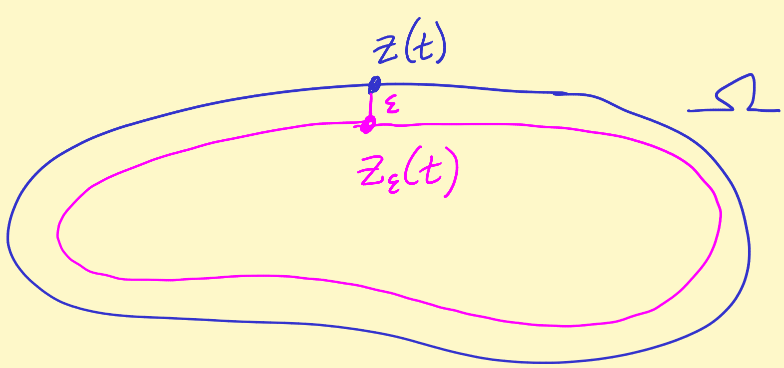
General case (Chap. 6)



Geometry fact:



There is a $\rho > 0$ such that when $\operatorname{dist}(z, b\Omega) < \rho$, there is a <u>unique</u> $z_0 \in b\Omega$ closest pt in $b\Omega$ to z. Picture holds: $z = z_0 + d \cdot N(z_0)$, $d = \operatorname{dist}(z, b\Omega)$. Pf: Implicit fcn thm, etc.



$$z_\varepsilon(t) = z(t) + \varepsilon\, i\, T(z(t))$$

Classic def^{n} of $H^2(b\Omega)$: $h \in \mathcal{O}(\Omega)$ and

$$\|h\|_\varepsilon^2 = \int_{b\Omega} |h(z_\varepsilon(t))|^2 \, |z_\varepsilon'(t)| \, dt$$

are uniformly bounded as $\varepsilon \searrow 0$.

Thm 6.1 If $h \in$ classical Hardy, $\exists$ $H \in H^2(b\Omega)$ (in book's sense) with

$$h(z) = (\mathcal{C}H)(z) = \frac{1}{2\pi i} \int_{b\Omega} \frac{H(w)}{w-z} \, dw$$

when $z \in \Omega$.

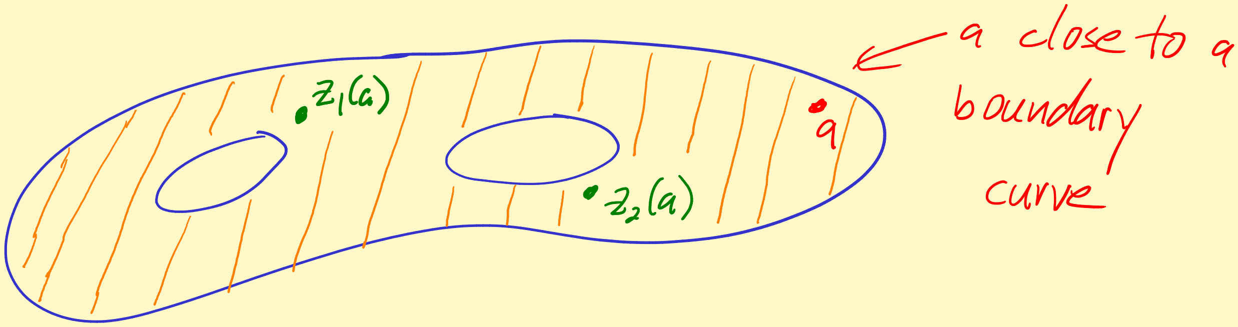
Pf: (Interesting part) How to get boundary values of h: $h_\varepsilon(t)$ are uniformly bounded in L^2 as $\varepsilon \searrow 0$. So can find a weakly convergent subseq $h_{\varepsilon_j} \to H$ meaning

$$\langle h_{\varepsilon_j}, v \rangle \xrightarrow[j\to\infty]{} \langle H, v \rangle \text{ for each } v \in L^2.$$

Take $v =$ Cauchy kernel C_z, $z \in \Omega$. etc.

Thm 6.2 Functions in $H^2(b\Omega) = \left(\text{closure of } A^\infty(\Omega) \text{ in } L^2(b\Omega)\right)$ satisfy classical defn.

What do zeroes of the Szegő kernel do when Ω has holes?



Fact Say Ω is n-connected (n boundary curves).
Given $a \in \Omega$, $S(z,a)$ has $n-1$ zeroes as a fcn of z in Ω counted with multiplicity (and no zeroes on $b\Omega$). And, as $a \to$ pt in a boundary curve, the zeroes become simple zeroes and head to seperate, different boundary curves! (as in picture)