Lecture 25 Miscellaneous

Ω bounded simply connected domain bounded by a C^∞-smooth Jordan curve.

Last week: Given $u \in C^\infty(b\Omega)$, the harmonic extension $\mathcal{E}u$ as a harmonic fcn on Ω with boundary values given by u, is equal to $h + \bar{H}$ where

$$h = \frac{P(S_a u)}{S_a} \quad \text{and} \quad H = \frac{P(L_a \bar{u})}{\underbrace{L_a}_{=0 \text{ at } a, \text{ the fixed pt in } \Omega.}}$$

Since S_a and L_a are non-vanishing, it follows that $\mathcal{E}u \in C^\infty(\bar{\Omega})$.

$$(\mathcal{E}u)(z) = \int_{b\Omega} \underbrace{p(z,w)}_{\text{Poisson kernel}} u(w)\, ds$$

get:
$$p(z,w) = \frac{S(z,w)S(w,a)}{S(z,a)} + \frac{\overline{S(z,w)L(w,a)}}{\overline{L(z,a)}}$$

Hmmm. a is arbitrary! Let $z=a$ in formula!

Get
$$p(a,w) = \frac{S(a,w)S(w,a)}{S(a,a)} + 0$$

$$= \frac{|S(w,a)|^2}{S(a,a)} \leftarrow \text{Poisson-Szegő kernel}$$

Thm: In a simply connected domain, the Poisson and Poisson-Szegő kernel coincide.

Remark Not true if Ω has holes.

h analytic: $\left\langle h, \frac{|S_a|^2}{S(a,a)} \right\rangle$

$$= \int_{b\Omega} h \, \overline{\frac{\bar{S}_a S_a}{S_a(a)}} \, ds$$

$$= \int (h S_a) \overline{\frac{S_a}{S_a(a)}} \, ds$$

$$= \left\langle \frac{h S_a}{S_a(a)}, S_a \right\rangle$$

reproduces analytic fcns at a

$$= \frac{h(a) S_a(a)}{S_a(a)} = h(a) \checkmark$$

Fact: P-S kernel is positive real valued kernel that reproduces analytic fcns just like S_a.

The case of real analytic bndry.

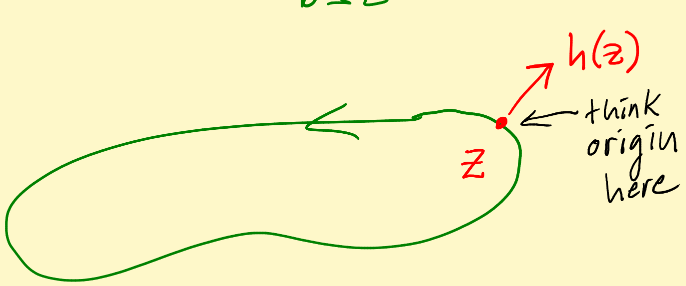
Proved carefully that $\mathcal{C}$ sends fcns that are real analytic on $b\Omega$ to analytic fcns on an open set containing $\overline{\Omega}$.

Follows that P has same property!

So S_a and L_a extend holomophically past the boundary. Can use the argument principal to count the zeroes of S_a and L_a.

MA 530 h meromorphic on a nbhd of $\overline{\Omega}$, and has no zeroes or poles on $b\Omega$.

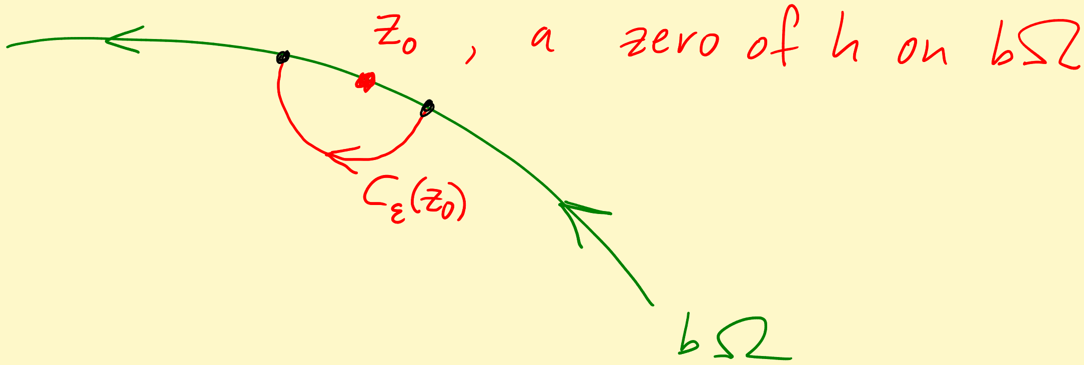
$$\frac{1}{2\pi i}\int_{b\Omega} \frac{h'(z)}{h(z)}\,dz = \underbrace{\frac{\Delta_{b\Omega} \arg h}{2\pi}}_{\substack{\#\text{ times} \\ h(z) \text{ spins}}} = \underbrace{(\#\text{ zeroes in }\Omega) - (\#\text{ poles})}_{\text{counted with mult}}$$



around origin as you go around $b\Omega$ counterclockwise

Fancier improved version If we allow zeroes on bndry, they count $\frac{1}{2}$ in Arg princ

Pf



$\Omega_\varepsilon = \Omega - D_\varepsilon(z_0)$ Do Arg princ on Ω_ε

Let $\varepsilon \to 0$. Say zero at z_0 has order m.

Then $\operatorname{Res}_{z_0} \frac{h'}{h} = m$

$$\frac{1}{2\pi} \Delta_{b\Omega} \arg h = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \Delta_{b\Omega_\varepsilon} \arg h$$

$$= \begin{pmatrix} \#\text{ zeroes inside} \\ \Omega \end{pmatrix} + \frac{1}{2} \begin{pmatrix} m \end{pmatrix}$$

$$- \begin{pmatrix} \#\text{ poles} \\ \text{inside} \end{pmatrix}$$

Application: $\overline{S_a} = \frac{1}{i} L_a T$

$$\arg \overline{S_a} = -\frac{\pi}{2} + \arg L_a + \arg T$$

Now take net increase as we travel around $b\Omega$:

$$\overline{\left[\left(\#\text{ zeroes } S_a \text{ in } \Omega\right) + \frac{1}{2}\left(\#\text{ zeroes } S_a \text{ in } b\Omega\right) \right]}$$

conjugate

$$= \left(\begin{matrix}\#\text{ zeroes } L_a \\ \text{in } \Omega\end{matrix}\right) + \frac{1}{2}\left(\begin{matrix}\#\text{zeroes} \\ L_a \text{ in } b\Omega\end{matrix}\right) - \underbrace{\left(\begin{matrix}\#\text{ poles } L_a \\ \text{in } \Omega\end{matrix}\right)}_{=1}$$

$$+ \frac{2\pi}{2\pi} \leftarrow \Delta_{b\Omega} \arg T$$

$$-\left[\geq 0 \right] = \left[\geq 0 \right]$$

Both must $= 0$! No zeroes inside or on $b\Omega$.

More fun with Poisson kernel.

$$\mathcal{E}u = h + \overline{H}$$

vanishes at a

Hmmm. Let u be real valued.

Then $\mathcal{E}u$ is real valued.

See h and $\overline{H}$ have same imaginary part.

They differ by a real constant.

So $\mathcal{E}u = h + \overline{H} = 2\,\mathrm{Re}\, h + c.$

Find c: Let $z = a$.

$$(\mathcal{E}u)(a) = h(a) + \underbrace{\overline{H(a)}}_{0} = 2\,\mathrm{Re}\, h(a) + c$$

$$c = -\mathrm{Re}\, h(a).$$

New formula: $(\mathcal{E}u) = 2\,\mathrm{Re}\, h - h(a)$

$$= 2\,\mathrm{Re}\, \frac{P(S_a u)}{S_a} - \frac{P(S_a u)(a)}{S(a,a)}$$

$$p(z,w) = 2\,\mathrm{Re}\left[\frac{S(z,w)\,S(w,a)}{S(z,a)}\right] - \frac{\overbrace{S(a,w)\,S(w,a)}^{=|S(w,a)|^2}}{S(a,a)}$$