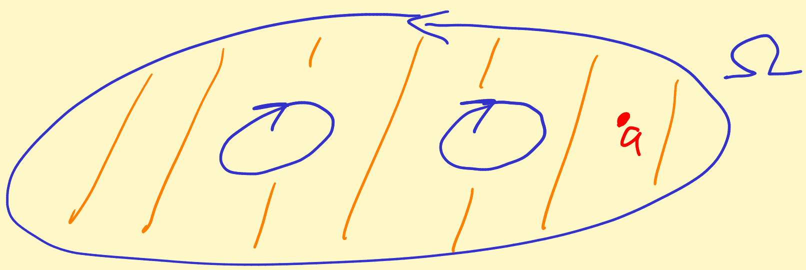
Lecture 26 The Bergman space

Szegő kernel on a multiply connected domain: book, chapters 9–14.

Pf that $L(z,a)$ is non-vanishing on $\bar{\Omega}-\{a\}$ in z variable works even when Ω has holes. Identity $\overline{S_a}=\frac{1}{i}L_a T$, which holds on $b\Omega$, shows that $S(z,a)\neq 0$ when $z\in b\Omega$, $a\in\Omega$. Note: $S(a,a)>0$.



Simply connected case: Riemann map $f_a(z)=\dfrac{S(z,a)}{L(z,a)}$

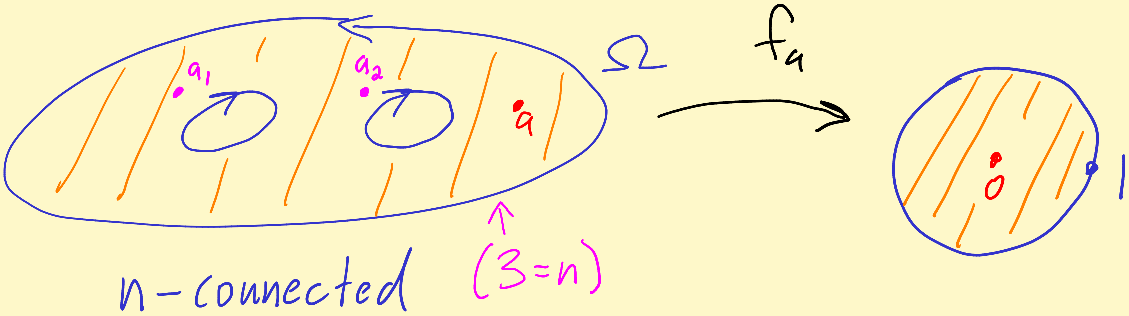
Hmmm. If Ω has holes: $\overline{S_a}=\frac{1}{i}L_a T$ ← take modulus

$$|S_a|=|L_a|\neq 0 \text{ on } b\Omega.$$

So define $f_a=\dfrac{S(z,a)}{L(z,a)}$ ← has a simple zero at a from simple pole of L_a and $S(a,a)>0$.

$$|f_a|=1 \text{ on } b\Omega.$$

Big thm: f_a is the Ahlfors map for Ω. (Think: a "Riemann map" for domains with holes.)



a_j zeroes of f_a

$a_0 = a$

Facts about f_a

1) f_a is the solution to the extremal problem: Among all holo fcns $h:\Omega \to D_1(0)$, f_a is the unique one with $|f_a'(a)|$ maximum and $f_a'(a) > 0$.

2) f_a maps Ω onto $D_1(0)$.

3) f_a maps Ω onto $D_1(0)$ n-times, counted with multiplicity. [$f(z_0) = w_0$ with mult m means $f(z) - w_0$ has a zero of mult m at $z = z_0$.]

Fancy word: f_a is an n-to-1 branched covering map of Ω onto $D_1(0)$.

Fascinating subject: Zeroes of $S(z,a)$ are the $(n-1)$ zeroes of the Ahlfors map f_a unequal to a.

Chap 15: Bergman space $H^2(\Omega)$

Def^n: $L^2(\Omega)$ complex valued L^2 fcns on Ω wrt Lebesgue area measure: $dx \wedge dy = dA$

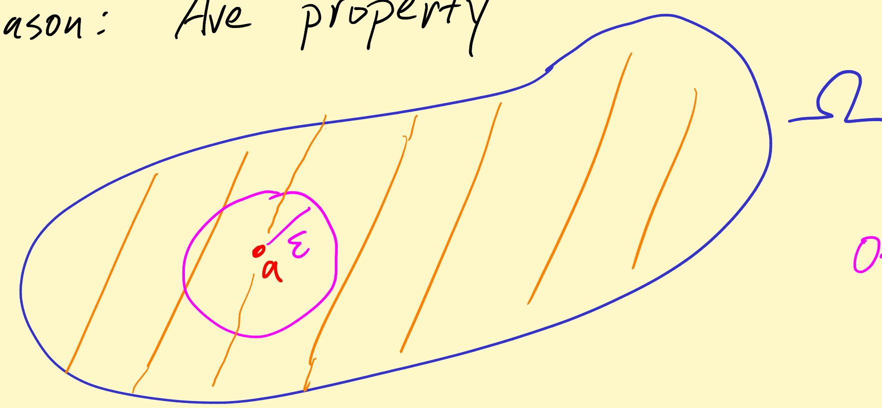
[Note: $dz \wedge d\bar{z} = -2i\, dx \wedge dy$

So $dA = \frac{i}{2} dz \wedge d\bar{z}$]

$H^2(\Omega)$ = holo fcns on Ω in $L^2(\Omega)$.

Fact $H^2(\Omega)$ is a closed supspace of $L^2(\Omega)$.

Easy reason: Ave property



$0 < \varepsilon < \mathrm{dist}_{b\Omega}(a)$

$$h(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a + re^{i\theta})\, d\theta \qquad 0 < r < \varepsilon$$

Next: integrate $r\,dr$, $0 \le r \le \varepsilon$.

$$h(a)\underbrace{\int_0^{\varepsilon} r\,dr}_{\frac{1}{2}\varepsilon^2} = \frac{1}{2\pi}\int_0^{\varepsilon}\int_0^{2\pi} h(a + re^{i\theta})\, d\theta\, r\,dr$$

Fact: Ave prop wrt area: $h(a) = \frac{1}{\pi\varepsilon^2}\int_{D_\varepsilon(a)} h\, dA$

Inner product $\langle u,v\rangle = \iint_\Omega u(z)\overline{v(z)}\,dA$

Sometimes $\langle u,v\rangle_{b\Omega}$, $\langle u,v\rangle_\Omega$ to distinguish.

Let $\chi_a^\varepsilon = \begin{cases} \frac{1}{\pi\varepsilon^2} & \text{in } D_\varepsilon(a) \\ 0 & \text{outside } D_\varepsilon(a) \end{cases}$

See $h(a) = \langle h, \chi_a^\varepsilon\rangle$

So $|h(a)| \le \|h\|\cdot\|\chi_a^\varepsilon\|$ $\qquad \|u\| = \langle u,u\rangle^{1/2}$

Thm: If holo $h_j \to u$ in $L^2(\Omega)$, then h_j converges uniformly on compact subsets to a holo fcn H. Measure theory $\Rightarrow$ $\underbrace{u = H \text{ a.e.}}_{\text{def}^{\text{n}} \text{ of a fcn in } H^2(\Omega).}$

Pf $h_j \to u$. So $\{h_j\}$ is a Cauchy seq in L^2.

so uniformly Cauchy in L^2

$$|h_j(a)| \le \frac{c}{\text{dist}_{b\Omega}(a)}\|h_j\|$$

$$\underbrace{|h_i(a) - h_j(a)|}_{\text{Uniformly Cauchy on compacts} \Rightarrow \text{conv unif on compacts.}} \le K\,\|h_i - h_j\|$$

Another relevant fact: If $h_j \to u$ in L^2, then $\exists$ subseq h_{j_k} that converges <u>pointwise</u> a.e.. See $u = H$ a.e.

<u>Thm</u>: $h \mapsto h(a)$ is a continuous linear fcnl on $H^2(\Omega)$. Riesz representation thm yields an element $K_a \in H^2(\Omega)$ such that

$$h(a) = \langle h, K_a \rangle \quad \text{for all } h \in H^2(\Omega)$$

Bergman kernel fcn

<u>Def^n</u> $K(z,a) = K_a(z)$. $\leftarrow$ holo in z

<u>Fact</u>: $K(a,b) = \overline{K(b,a)}$

<u>Pf</u>:
$$\begin{array}{ccccc} \langle K_b, K_a \rangle & = & K_b(a) & = & K(a,b) \\ \| & & & & \\ \overline{\langle K_a, K_b \rangle} & = & \overline{K_a(b)} & = & \overline{K(b,a)} \end{array} \checkmark$$

$K(z,w)$ holo in z, antiholo in w

Next time: In s.c. domain: $f_a = c K_a$, $\quad c = \sqrt{\frac{\pi}{K(a,a)}}$