Lecture 27 The Bergman kernel

Suppose Ω is a bounded domain. The Bergman space $H^2(\Omega)$ is the (closed) subspace of $L^2(\Omega)$ given by holo fcns on Ω in $L^2(\Omega)$. Since $H^2(\Omega) \subset L^2(\Omega)$ is a closed subspace, the orthogonal projection $B: L^2(\Omega) \to H^2(\Omega)$ exists (and is a bounded operator).

Def^n: $Bu = h$ means: for $g \in H^2(\Omega)$, $u \in L^2(\Omega)$:

$$g \mapsto \langle g, u \rangle \quad \text{is a cont. lin fcn}^{\ell} \text{ on } H^2(\Omega).$$

Riesz rep thm $\Rightarrow \exists\, h \in H^2(\Omega)$ so that

$$\langle g, u \rangle = \langle g, \underset{\substack{\uparrow \\ h = Bu}}{h} \rangle \quad \text{for all } g \in H^2(\Omega)$$

Fact The Bergman kernel is the kernel for the Berg proj.

Why: Above:

$$\langle g, u \rangle = \langle g, Bu \rangle$$

$$\langle u, g \rangle = \langle Bu, g \rangle$$

let $g = K_a$

$$\langle u, K_a \rangle = \langle Bu, K_a \rangle = (Bu)(a)$$

So $$(Bu)(a) = \langle u, K_a \rangle = \iint_\Omega u(z)\, \overline{K_a(z)}\, dA$$

$$\boxed{(Bu)(a) = \iint_{z\in\Omega} K(a,z)\, u(z)\, dA_z}$$

since $\quad K(z,a) = K_a(z) \quad$ and $\quad K(a,z) = \overline{K(z,a)}$.

Lucky connection between $L^2(\Omega)$ and holo fcns.

C-R eqns:

$$f' = \begin{cases} \underline{u_x + i v_x} \\ \\ v_y - i u_y \end{cases} \qquad (f = u + iv)$$

Jacobian matrix: $\quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix}$

$$J = \begin{bmatrix} u_x & \overset{-v_x}{u_y} \\ v_x & \underset{u_x}{v_y} \end{bmatrix}$$

$$\det J = (u_x)^2 + (v_x)^2 = |f'|^2$$

2-D change of vars formula: $\quad \Phi : \Omega_1 \xrightarrow[\text{onto}]{1\text{-}1} \Omega_2$ diffeo

$$\iint_{\Omega_1} |\det J_\Phi|\,(u\circ\Phi)\, dA = \iint_{\Omega_2} u\, dA$$

If Φ comes from a biholo mapping:

$$\iint_{\Omega_1} |f'|^2 (u \circ f)\, dA = \iint_{\Omega_2} u\, dA$$

Fact : $u \longmapsto \underbrace{f' \cdot (u \circ f)}_{\Lambda u}$ is an isometry of $L^2(\Omega_2) \longrightarrow L^2(\Omega_1)$ and preserves Bergman spaces too!

Why: $$\iint_{\Omega_2} u \bar{v}\, dA = \iint_{\Omega_1} \underbrace{|f'|^2}_{f'\overline{f'}} (u \circ f)\, \overline{(v \circ f)}\, dA$$

$$\boxed{\langle u, v \rangle_{\Omega_2} = \langle \Lambda u, \Lambda v \rangle_{\Omega_1}}$$

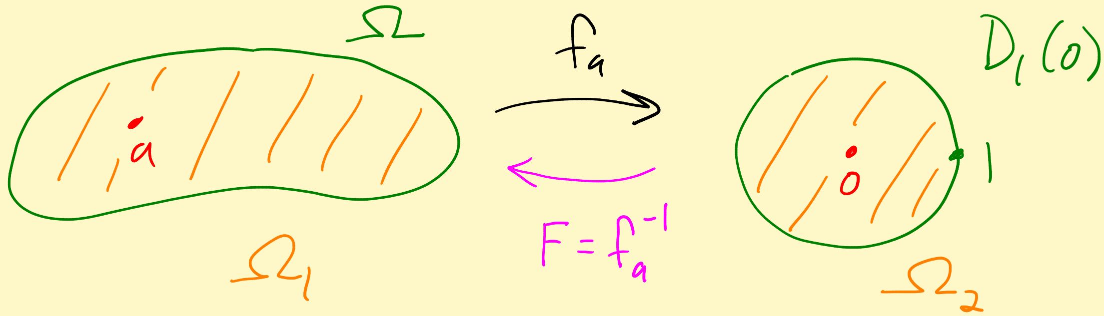
Let $u = v$: $\quad \|u\|_{\Omega_2} = \|\Lambda u\|_{\Omega_1}$

Exciting fact Ω bounded and simply connected, $a \in \Omega$. The Rieman map satisfies

$$f_a' = c K_a \quad \text{where} \quad c = \sqrt{\frac{\pi}{K(a,a)}}$$

Pf: Hmmm. Given $h \in H^2(\Omega)$, calculate

$$\langle h, f_a' \rangle \leftarrow \text{hope } = c\, h(a)$$



$$F(f_a(z)) = z$$

$$\underbrace{F'(f_a(z))\, f_a'(z)}\ = 1 \quad \leftarrow \text{inverse fcn thm, chain rule}$$

$$\langle h, f_a' \rangle = \iint_\Omega h \cdot \overline{f_a'}\ \underbrace{f_a' \cdot (F' \circ f_a)}_{=1}\ dA$$

$$= \iint_\Omega |f_a'|^2 \left(h \circ F \circ f_a \right) (F' \circ f_a)\ dA$$

$$\underset{\substack{\uparrow \\ \text{change vars} \\ \text{formula}}}{=} \iint_{D_1(0)} (h \circ F)\, F'\ dA \underset{\substack{\uparrow \\ \text{ave} \\ \text{prop!}}}{=} \pi \cdot 1^2 \,(h(\underbrace{F(0)}_{a}))\, F'(0)$$

$$= \pi\, F'(0)\, h(a)$$

Aha! $\quad f_a' = \pi \overline{F'(0)}\ K_a \qquad \left(F'(0) = \frac{1}{f_a'(a)} > 0 \right)$

Play around: let $z = a$, $\qquad f_a'(a) = \pi \dfrac{1}{f_a'(a)} K(a,a)$

Get $|f_a'(a)|^2 = \pi K(a,a)$ and

$$f_a' = \sqrt{\frac{\pi}{K(a,a)}}\, K_a \quad \checkmark$$

Observation $\quad f : \Omega_1 \to \Omega_2 \quad$ biholo, $\quad F = f^{-1}$

$$\Lambda_1 u = f' \cdot (u \circ f) \qquad \Lambda_1 : L^2(\Omega_1) \leftrightarrow L^2(\Omega_2)$$

$$\Lambda_2 U = F' \cdot (U \circ F) \qquad \Lambda_2 : L^2(\Omega_2) \leftrightarrow L^2(\Omega_1)$$

$$\boxed{\Lambda_2 = \Lambda_1^{-1}}$$

Calculation above showed

$$\langle \Lambda_1 u, v \rangle_{\Omega_2} = \langle u, \Lambda_2 v \rangle_{\Omega_1}$$

Transformation formula for Bergman kernels under conformal maps :

$$\boxed{f'(z)\, K_2(f(z), w) = K_1(z, F(w))\, \overline{F'(w)}}$$

or $$f'(z)\, K_2(f(z), f(\zeta))\, \overline{f'(\zeta)} = K_1(z,\zeta)$$

just let $\zeta = F(w)$ in box.
Use chain rule.

Case $\Omega_2 = D_1(0)$. Know B. kernel for $D_1(0)$ is

$$\underbrace{K(z,w)}_{\text{Berg kernel for } D_1(0)} = \frac{1}{\pi(1-z\bar{w})^2}$$