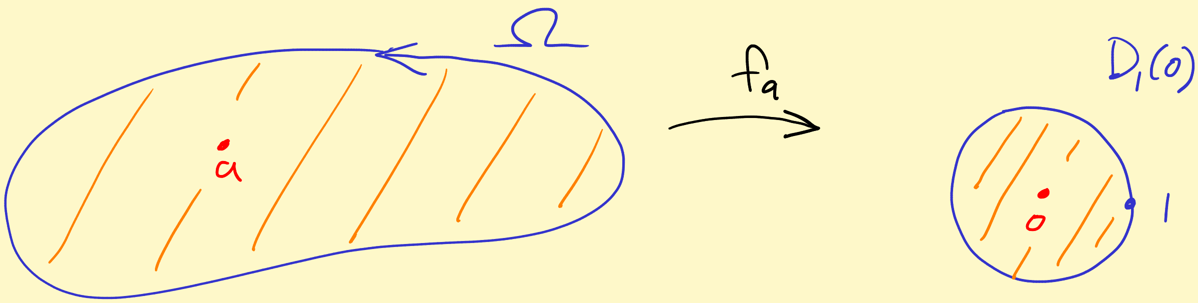
Lecture 28 The Bergman projection

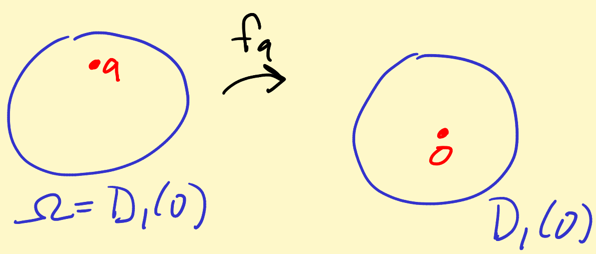


Riemann map: $f_a'(z) = c\, K_a(z)$, $c = \sqrt{\dfrac{\pi}{K(a,a)}}$

Note: $K(a,a) = K_a(a) = \langle K_a, K_a \rangle = \| K_a \|^2$

$\left[\, h(a) = \langle h, K_a \rangle : \text{ so } K_a \not\equiv 0. \right.$ (Take $h \equiv 1$.) $\left.\right]$

Bergman kernel for $D_1(0)$:



Riemann map is $f_a(z) = \varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$

Note: $\varphi_a'(z) = \dfrac{1-|a|^2}{(1-\bar{a}z)^2}$ and $\varphi_a'(a) = \dfrac{1}{1-|a|^2} > 0$

$$f_a'(z) = c\, K_a(z) = c\, K(z,a)$$

$$\frac{1-|a|^2}{(1-\bar{a}z)^2} = c\, K(z,a)$$

How to find c: Easy fact from ave princ.

$$K_0(z) = \frac{1}{\pi} \quad \text{because} \quad h(0) = \frac{1}{\pi \cdot 1^2} \iint_{D_1(0)} h \, dA$$

$$= \left\langle h, \frac{1}{\pi} \right\rangle, \quad \forall h \in H^2(D_1(0))$$

$$\boxed{\frac{1-|a|^2}{(1-\bar{a}z)^2} = c\,K(z,a)} \leftarrow \text{plug in } z=0$$

$$1-|a|^2 = c\,K(0,a) = c\,\overline{K(a,0)} = c \cdot \frac{1}{\pi} \qquad \boxed{c = \pi(1-|a|^2)}$$

$$\boxed{K(z,a) = \frac{1}{\pi(1-\bar{a}z)^2}} \quad \text{Unit disc.}$$

History. Stefan Bergman: Orthonormalize z^n, $n=0,1,2,\ldots$ over $D_1(0)$ wrt area.

Easy to check $z^n \perp z^m$ if $n \neq m$. ✓

$$\|z^n\|^2 = \iint_{D_1(0)} |z^n|^2 \, dA = \int_0^{2\pi} \int_0^1 r^{2n} \, r\,dr\,d\theta$$

$$= 2\pi \left. \frac{r^{2n+2}}{2n+2} \right]_0^1 = \frac{\pi}{n+1}$$

$$\|z^n\| = \sqrt{\frac{\pi}{n+1}}$$

Orthonormal basis for $H^2(D_1(0))$: $\left\{ \dfrac{z^n}{\sqrt{\frac{\pi}{n+1}}} \right\}_{n=0}^{\infty} \leftarrow \varphi_n$

Bergman: $K(z,w) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(w)}$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1) z^n \bar{w}^n$$

Use $\frac{1}{1-r} = 1 + r + r^2 + \cdots$

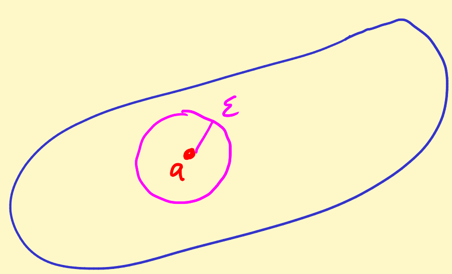
$$\frac{1}{(1-r)^2} = 1 + 2r + 3r^2 + \cdots$$

Let $r = z\bar{w}$. Get same formula.

Fun fact $\quad K_a = B\varphi_a$ where $\varphi_a \in C_0^\infty(\Omega)$.

Hmmm. If we knew $B: C^\infty(\bar{\Omega}) \circlearrowleft$, then we'd get that $K_a \in C^\infty(\bar{\Omega})$. Deduce R. Map $f_a' \in C^\infty(\bar{\Omega})$, so $f_a \in C^\infty(\bar{\Omega})$

Claim Let φ_a be a radially symmetric C_0^∞ bump fcn centered at a.



Ω

$\varphi_a \in C_0^\infty(D_\varepsilon(a))$ extend to $\mathbb{C}$.

Claim: $h(a) = \langle h, \varphi_a \rangle$ because

$$\langle h, \varphi_a \rangle = \iint h(a+z)\, \underset{\substack{\uparrow \\ \text{real valued}}}{\varphi_a(a+z)}\, dA$$

$$= \iint h(a+re^{i\theta})\, \underbrace{\varphi_a(a+re^{i\theta})}_{\psi(r)}\, r\,dr\,d\theta$$

$$= \int_0^{\varepsilon} \underbrace{\int_0^{2\pi} h(a+re^{i\theta})\, d\theta}_{\substack{2\pi\, h(a) \\ \text{ave prop}}}\ \psi(r)\, r\,dr$$

$$= h(a) \underbrace{\left[2\pi \int_0^{\varepsilon} \psi(r)\, r\,dr \right]}_{\text{unit mass of bump fcn}} \checkmark$$

Consequently $\underset{\substack{\| \\ h(a)}}{\langle h, \varphi_a \rangle} = \langle h, B\varphi_a \rangle$ for all $h \in H^2(\Omega)$.

$\uparrow$ So $K_a = B\varphi_a$

Goal: $\quad Bu = u - \frac{\partial}{\partial z} 4\, G\left(\frac{\partial u}{\partial \bar{z}}\right)$

G Green's operator : $\quad \Delta(Gv) = v$

$$Gv\big|_{b\Omega} = 0$$

Know $G : C^\infty(\bar{\Omega})$ ↺. Hence so does B.

Orthogonal complement of $H^2(\Omega)$ in $L^2(\Omega)$

Lemma Suppose Ω is a smooth, bounded domain and $\varphi \in C^\infty(\bar{\Omega})$ and $\varphi\big|_{b\Omega} = 0$. Then $\langle \frac{\partial \varphi}{\partial z}, h \rangle = 0$ for all $h \in H^2(\Omega)$.

Pf: Easy if $h \in \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})$. Then

$$\langle \frac{\partial \varphi}{\partial z}, h \rangle = \iint_\Omega \frac{\partial \varphi}{\partial z} \cdot \bar{h} \left(\underbrace{\frac{i}{2} dz \wedge d\bar{z}}_{dA = dx \wedge dy} \right)$$

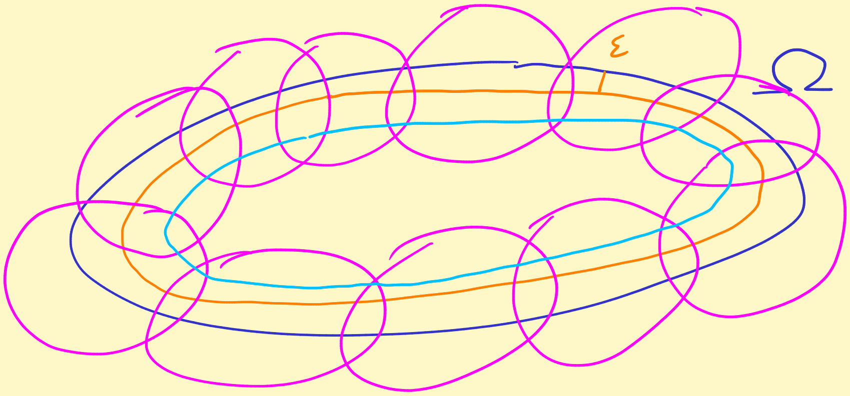
$$= \frac{i}{2} \iint_\Omega \frac{\partial}{\partial z} \left[\varphi \cdot \bar{h} \right] dz \wedge d\bar{z}$$

Note: $\frac{\partial \bar{h}}{\partial z} = 0$

$$\underset{\text{Green's}}{=} \frac{i}{2} \int_{b\Omega} \varphi \cdot \bar{h} \; d\bar{z} = 0$$

$\varphi \equiv 0$ on $b\Omega$.

Classic trick: If $h \in H^2(\Omega)$:

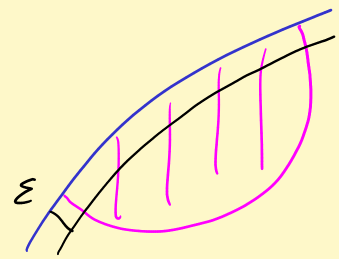


are a locally finite open cover of $\overline{\Omega}$

$\exists$ C_0^∞ partition of unity $\{\psi_j\}_{j=0}^N$ subordinate to cover.

$$\left\langle \frac{\partial \varphi}{\partial z}, h \right\rangle = \left\langle \sum_{j=1}^{N} \frac{\partial(\varphi \psi_j)}{\partial z}, h \right\rangle$$

Reduce problem to simple case



Slide φ in a distance ε. Let $\varepsilon \searrow 0$