Lecture 29 The Bergman projection & the Green's operator

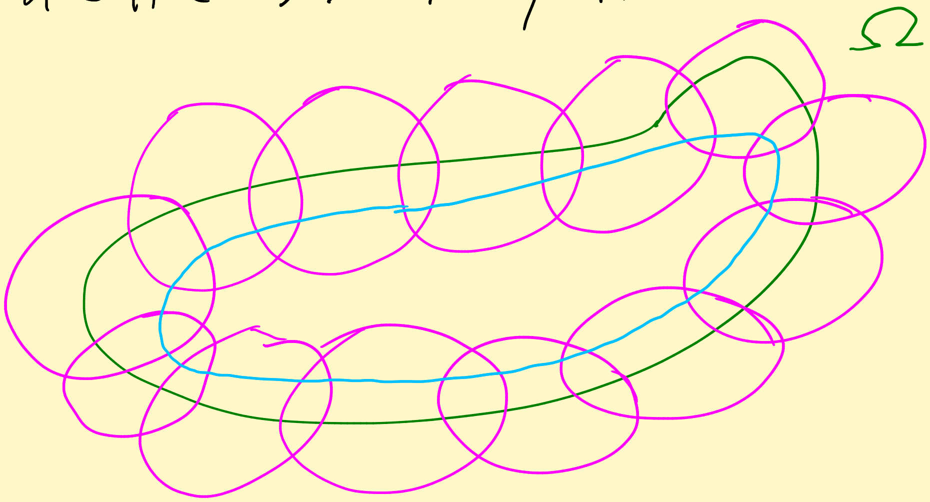
Thm: Ω smooth bnd domain. Then $\frac{\partial \varphi}{\partial z} \perp H^2(\Omega)$ in $L^2(\Omega)$ when $\varphi \in C^\infty(\overline{\Omega})$ and $\varphi|_{b\Omega} \equiv 0$.

Pf: Easy to show $\langle h, \frac{\partial \varphi}{\partial z} \rangle = 0$ when $h \in A^\infty(\Omega)$.

Def^n: $A^\infty(\Omega) = \mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega})$.

Pf last time via Green's identity.

Suppose $h \in H^2(\Omega)$. Handy trick

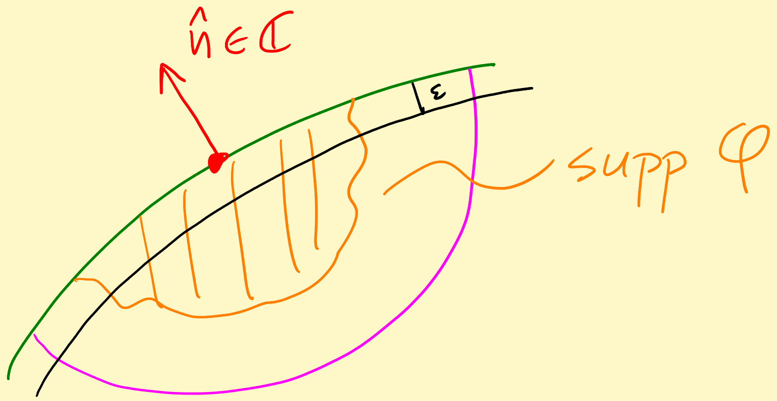


D_j's, U open cover of $\overline{\Omega}$. Get a C_0^∞ part. of unity subordinate to cover. $\{\psi_j\}_{j=0}^N \leftarrow \psi_0 \in C_0^\infty(U)$.

So $\sum_{j=0}^N \psi_j \equiv 1$ on $\overline{\Omega}$.

$$\varphi = \sum_{j=0}^N (\varphi \psi_j)$$

Reduce $\perp$ question to this



$\varphi \sim \varphi \psi_j$

Trick: Consider $V_\varepsilon = \begin{cases} \frac{\partial \varphi}{\partial \bar{z}}(z - \varepsilon \hat{n}) \\ 0 \text{ between } b\Omega \text{ and } b\Omega - \varepsilon \hat{n} \end{cases}$

Clear that 1. $\langle h, V_\varepsilon \rangle = 0$,

2. $\langle h, \frac{\partial \varphi}{\partial \bar{z}} \rangle = \lim_{\varepsilon_k \downarrow 0} \langle h, V_{\varepsilon_k} \rangle = 0$

Thm Ω smooth. $Bu = u - \frac{\partial}{\partial z} 4\, G\left(\frac{\partial u}{\partial \bar{z}}\right)$

when $u \in C^\infty(\bar{\Omega})$.

Pf Recall: $\begin{cases} \Delta(Gv) = v \text{ on } \Omega \\ Gv\big|_{b\Omega} \equiv 0. \end{cases}$

$$\Delta v = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \quad = 4\left[\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]$$ ✓

Regularity properties of D. prob : $\Delta\varphi = 0$ in Ω

$\varphi|_{b\Omega}$ = something

$\Rightarrow \quad G : C^\infty(\bar{\Omega}) \circlearrowleft$

Remark Proved using Szegő projection in simply connected case. Also true when Ω has holes.

See Chap 6–14 for details.

Pf: $u \in C^\infty(\bar{\Omega})$. First compute

$$\frac{\partial}{\partial \bar{z}}\Big[\underbrace{u - \frac{\partial}{\partial z} 4\, G\Big(\frac{\partial u}{\partial \bar{z}}\Big)}_{h}\Big]$$

$$= \frac{\partial u}{\partial \bar{z}} - \underbrace{4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}}_{\Delta} G\Big(\frac{\partial u}{\partial \bar{z}}\Big)$$

$$= \frac{\partial u}{\partial \bar{z}} - \frac{\partial u}{\partial \bar{z}} \equiv 0 \quad !$$

So h is holo.

Next: Let $\varphi = 4\, G\Big(\frac{\partial u}{\partial \bar{z}}\Big) \in C^\infty(\bar{\Omega})$

and $\varphi|_{b\Omega} \equiv 0$.

$$u = h - \frac{\partial \varphi}{\partial z}$$

$\frac{\partial \varphi}{\partial z} \perp H^2(\Omega)$ in $L^2(\Omega)$.

Apply Berg proj B:

$$Bu = \underbrace{Bh}_{=h} - \underbrace{B\left(\frac{\partial \varphi}{\partial z}\right)}_{\equiv 0}$$

So $h = Bu$.

<u>Cor</u> $B: C^\infty(\bar{\Omega}) \circlearrowleft$

<u>Cor</u> In simply connected case $f_a' = cK_a = cB\varphi_a$

φ_a: C_0^∞ bump fcn

So Riemann map is in $C^\infty(\bar{\Omega})$.

<u>History</u> Painlevé proved this before Carathéodory proved continuous extension!

<u>Cor</u> $A^\infty(\Omega)$ is <u>dense</u> in $H^2(\Omega)$.

<u>Pf</u>: Given $h \in H^2(\Omega)$.

Step 1 $\quad h_\varepsilon = \chi_\varepsilon \cdot h$ where

$$\chi_\varepsilon(z) = \begin{cases} 1 & \text{if } \operatorname{dist}(z, b\Omega) \geq \varepsilon, \quad z \in \Omega \\ 0 & \text{if } \operatorname{dist}(z, b\Omega) < \varepsilon \end{cases}$$

Easy to show $h_{\varepsilon_k} \xrightarrow{L^2} h$ as $\varepsilon_k \searrow 0$.

Sketch: $|\langle h, \underset{\chi_\Omega - \chi_\varepsilon}{\psi} \rangle| \leq \|h\| \underbrace{\|\psi\|}_{\sim\sqrt{\varepsilon \cdot \ell}}$ $\quad \ell =$ arc length

Step 2: Smooth out χ_{ε_k} with C_0^∞ bump fcn.

$$\varphi_\delta * (\chi_{\varepsilon_k} h), \quad 0 < \delta < \frac{\varepsilon_k}{3}.$$

Know $\varphi_\delta * (\chi_{\varepsilon_k} h) \longrightarrow \chi_{\varepsilon_k} h$ in L^2.

Get seq $\varphi_j \in C_0^\infty(\Omega)$ with $\varphi_j \xrightarrow{L^2} h$.

Finally $\underbrace{B\varphi_j}_{\in A^\infty(\Omega)} \xrightarrow{L^2} Bh = h$

Interesting dense subspace of $H^2(\Omega)^\perp$ in $L^2(\Omega)$:

$$\left\{ \frac{\partial \varphi}{\partial z} : \varphi \in C_0^\infty(\Omega) \right\} \text{ is dense.}$$

Pf: Know such fcns $\frac{\partial\varphi}{\partial z}$, $\varphi \in C_0^\infty(\Omega)$ are $\perp$ to $H^2(\Omega)$ (because $\varphi\big|_{b\Omega} = 0$).

Suppose $v \in L^2(\Omega)$ and $v \perp \frac{\partial\varphi}{\partial z}$ $\forall\, \varphi \in C_0^\infty$.

Then
$$0 = \left\langle v, \frac{\partial\varphi}{\partial z} \right\rangle = \iint_\Omega v\, \overline{\frac{\partial\varphi}{\partial z}}\, dA$$
$$= \iint_\Omega v\, \frac{\partial\bar{\varphi}}{\partial\bar{z}}\, dA$$

for all $\bar{\varphi} \in C_0^\infty(\Omega)$.

Aha! v is a weak sol^n to $\frac{\partial v}{\partial\bar{z}} = 0$ in L^2.

Weyl's lemma $\Rightarrow$ $v = h$ a.e. where $h \in \mathcal{O}(\Omega)$.

Consequently, $v \in H^2(\Omega)$.

Fun facts In simply connected case

1. $K(z,a) = 4\pi\, S(z,a)^2$

2. $f_a'(z) = c\, K(z,a)$ $\quad c = \sqrt{\frac{\pi}{K(a,a)}}$

3. $f_a(z) = \frac{S(z,a)}{L(z,a)}$

Key to 1. $L(z,a) = \frac{1}{2\pi} \frac{1}{(z-a)} + H_a(z)$

$H_a \in A^\infty(\Omega)$.

$L(z,a)^2$ has a <u>double</u> pole at $z=a$.

<u>Fact</u> $\mathrm{Res}_a \, L(z,a)^2 = 0$