Lecture 30 More properties of the kernels

Thm: In a smooth simply connected domain Ω

$$K(z,a) = 4\pi\, S(z,a)^2$$

Pf: Let $G = K_a - 4\pi S_a^2$. Show $G \equiv 0$.

Know $G \in A^\infty(\Omega) \subset L^2(b\Omega)$.

Claim $\bar{G}\bar{T}$ is orthogonal in $L^2(b\Omega)$ to the Hardy space $H^2(b\Omega)$ and conjugates of fcns in $H^2(b\Omega)$.

Lemma: $u \in L^2(b\Omega)$ and $u \perp H^2(b\Omega)$ and

$$u \perp \{\bar{h} : h \in H^2(b\Omega)\} \implies u \equiv 0.$$

Pf: Next time. (Good thing to try yourself.)

Pf of claim: Know $\bar{G}\bar{T} \perp H^2(b\Omega)$.

Suppose $H \in H^2(b\Omega)$. Need to show $\bar{G}\bar{T} \perp \bar{H}$.

Step 1 $\langle \bar{K}_a \bar{T}, \bar{H} \rangle_{b\Omega}$

$$= \int_{b\Omega} \bar{K}_a H \underbrace{\{\bar{T}\, ds\}}_{d\bar{z}}$$

$$= \int_{b\Omega} H\,\overline{K_a}\,d\bar{z}$$

$$\underset{\text{Green's}}{=} \iint_{\Omega} \underbrace{\frac{\partial}{\partial z}\left[H\,\overline{K_a}\right]}_{H'\,\overline{K_a}}\ \underbrace{dz \wedge d\bar{z}}_{-2i\,dx\wedge dy}$$

$$= -2i \langle H', K_a \rangle_{\Omega}$$

$$= -2i\, H'(a)$$

Next $\langle \overline{4\pi S_a^2}\,\overline{T}, \overline{H} \rangle_{b\Omega}$:

Use identity $\overline{S_a} = \frac{1}{i} L_a T$

Square: $\overline{S_a}^2 = -L_a^2 T^2$

Multiply by $\overline{T}$: $\boxed{\overline{S_a^2}\,\overline{T} = -L_a^2\,T}$

<u>Lemma</u>: Principal part of L_a^2 at $z=a$ is

$$\frac{1}{4\pi^2}\frac{1}{(z-a)^2}$$ ← surprising thing: no residue term.

Red box

$$\int_{b\Omega} L_a^2 \, dz = 2\pi i \operatorname{Res}_a L_a^2$$

$$\Vert$$

$$\int_{b\Omega} L_a^2 \underbrace{T \, ds}_{dz} = \int_{b\Omega} -\overline{S_a^2}\,\overline{T}$$

$$= -\overline{\int_{b\Omega} S_a^2 \, dz} = 0$$

by Cauchy's thm.

Back to

$$\langle \overline{4\pi S_a^2}\,\overline{T}, \overline{H} \rangle_{b\Omega} = 4\pi \int_{b\Omega} \overline{S_a^2}\,\overline{T} H \, ds$$

$$= (4\pi)\left(\int_{b\Omega} -L_a^2 T H \, ds \right)$$

$$= -4\pi \int_{b\Omega} L_a^2 H \, dz$$

$$= -4\pi \left[2\pi i \underbrace{\operatorname{Res}_a \left(L_a^2 H \right)}_{\frac{1}{4\pi^2} H'(a)} \right] = -2i H'(a)$$

same!

They cancel when subtracted

So $\overline{GT} \perp \overline{H}$ too. Done!

More consequences of red box idea:

<u>Fact</u>: $L(b,a) = -L(a,b)$

<u>why</u>: $\begin{cases} \overline{S_a} = \frac{1}{i} L_a T \\ \overline{S_b} = \frac{1}{i} L_b T \end{cases}$

$$\Longrightarrow \overline{S_a}\,\overline{S_b}\,\overline{T} = -L_a L_b T$$

So $$\int_{b\Omega} \overline{S_a S_b \underbrace{T\, ds}_{T ds = dz}} = -\int_{b\Omega} L_a L_b \underbrace{T\, ds}_{dz}$$

$$0 \underset{\substack{\uparrow \\ \text{Cauchy's thm}}}{=} \overline{\int_{b\Omega} S_a S_b\, dz} = -\int_{b\Omega} L_a L_b\, dz$$

$$= -2\pi i \left(\operatorname{Res}_a L_a L_b + \operatorname{Res}_b L_a L_b \right)$$

$$= -2\pi i \left(\frac{L(a,b)}{2\pi} + \frac{L(b,a)}{2\pi} \right)$$

So $L(a,b) + L(b,a) = 0$. ✓

Proved when $a \neq b$. [Don't care about $a=b$ at pole!]

<u>Fact</u>: $L(z,a) \neq 0$ when $z \in \Omega - \{a\}$.

Proof for Ω s.c.

<u>Pf</u>: Hmmm. If $L(b,a) = 0$, $b \in \Omega - \{a\}$,

then $L(a,b) = 0$ too.

So $L(z,a)\,L(z,b)$ has removabe sing

at a and b because zero of $L(z,b)$ at

$z=a$ cancels simple pole of $L(z,a)$ at a,

and similarly for pole of $L(z,b)$ at b.

$$\text{Get } \underbrace{\overline{S_a S_b T}}_{\perp H^2(b\Omega)} = - \underbrace{L_a L_b}_{\in A^\infty(\Omega)} T$$

$\perp \bar{h}$ when $h \in H^2(b\Omega)$

Use Lemma: See $S_a S_b \equiv 0$. ↯

[because $s_a s_b \equiv 0$ on $\Omega \Rightarrow s_a \equiv 0$ or $s_b \equiv 0$

and that can't be (because $h(a) = \langle h, s_a \rangle_{b\Omega}$

when $h \equiv 1$)]

Remarks about $Bu = u - \frac{\partial}{\partial z}\left[4G\left(\frac{\partial u}{\partial \bar{z}}\right)\right]$ ← Spencer's formula

Interesting thing: Pf showed that

$$4\frac{\partial}{\partial z}G(v)$$

solves a $\bar{\partial}$-problem: Let $u = 4\frac{\partial}{\partial z}G(v)$. Then

$$\frac{\partial u}{\partial \bar{z}} = \underbrace{4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}}_{\Delta} G(v) = v$$

And the solution u is special because it

is $\perp$ analytic fcns $u = \frac{\partial}{\partial z}\big[\underbrace{4Gv}_{=0 \text{ on } b\Omega}\big]$

$\left(u = \frac{\partial \varphi}{\partial z}, \varphi|_{b\Omega} \equiv 0 \perp \text{ Bergman space in } L^2(\Omega)\right).$