Lecture 31 Not so good proofs v.s. good proofs

Remark: Last time $u = 4\frac{\partial}{\partial z}G(v)$ solves $\frac{\partial u}{\partial \bar{z}} = v$ in a smooth s.c. domain with $u \perp H^2(\Omega)$.

Note: u is the unique sol^n with $u \perp H^2(\Omega)$.

why: If u_1 and u_2 both did, then

$$\frac{\partial}{\partial \bar{z}}(u_1 - u_2) = v - v \equiv 0$$

So $u_1 - u_2$ is analytic, and therefore in $H^2(\Omega)$. [We assume $v \in C^\infty(\bar{\Omega})$.]. And $u_1 - u_2 \perp H^2(\Omega)$. So $u_1 - u_2 = 0$ in $L^2(\Omega)$, and $u_1 = u_2$ as fcns in $C^\infty(\bar{\Omega})$.

Remark: In SCV, $B = I - \vartheta N \bar{\partial}$

$$\bar{\partial} u = \frac{\partial u}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial u}{\partial \bar{z}_n} d\bar{z}_n$$

ϑ = formal adjoint of $\bar{\partial}$-operator in L^2.

N = $\bar{\partial}$-Neumann operator

Key ingredient in Fefferman's thm about extension of biholo maps:

Bergman kernels and projections transform:

$f : \Omega_1 \to \Omega_2$ biholo

In $\mathbb{C}$: $K_1(z,w) = f'(z) K_2(f(z), f(w)) \overline{f'(w)}$

In $\mathbb{C}^n$: Same formula! with $f'(z)$ replaced by holo jacobian determinant

$$u = \det \left[\frac{\partial f_i}{\partial z_j} \right]_{i,j=1,\dots,n}.$$

<u>Pf</u>: $\Lambda : \varphi \mapsto u(\varphi \circ f)$

is an isometry of L^2 preserving the Bergman space. Same pf via

$|u|^2 =$ det Jacobian as a map $\mathbb{R}^{2n} \circlearrowleft$

Research advice: Play around with multiple proofs of a result until you find <u>the</u> one.

<u>Last time</u>: $G = K_a - 4\pi S_a^2$ in a s.c. smooth domain. Then $G \in A^\infty(\Omega)$.

Showed $\overline{G}\overline{T}$ is $\perp_{b\Omega}$ $H^2(b\Omega)$ and $\overline{H^2}(b\Omega)$.

conjugates

And so $\overline{G}\overline{T} \equiv 0 \Rightarrow G \equiv 0$ and $K_a = 4\pi S_a^2$.

Recall $\perp$ decomposition $u = h + \overline{H}\overline{T}$, $h, H \in H^2(b\Omega)$.

Easy part: $\overline{G}\overline{T} \perp H^2(b\Omega)$.

Next, showed that $\overline{G}\overline{T} \perp \overline{H}$ when $H \in H^2(b\Omega)$.

$$0 = \langle \overline{G}\overline{T}, \overline{H} \rangle = \int_{b\Omega} \overline{G}\,\overline{T}\, H \, ds$$

$$= \int_{b\Omega} \overline{G}\, H \, d\bar{z}$$

$$\underset{\text{Green's}}{=} \iint_{\Omega} \overline{G}\, H' \, \underbrace{dz \wedge d\bar{z}}_{-2i\, dx \wedge dy}$$

Hmmm. $0 = \langle H', 2i\, G \rangle$

for all $H \in A^\infty(\Omega)$!

Aha! In a simply connected domain, G

has an analytic antiderivative H with

$H' = G \in A^\infty(\Omega)$. So $H \in A^\infty(\Omega)$.

$$0 = \langle \underbrace{H'}_{=G}, 2iG \rangle$$

$$0 = 2\|G\|^2_{L^2(\Omega)}$$

$\Longrightarrow$ $G \equiv 0$. ✓

Thm: Suppose Ω is a smooth s.c. domain. If $u \perp H^2(b\Omega)$ and $\perp \overline{H^2}(b\Omega)$, then $u = 0$.

Bad proof # 2

$u \perp H^2(b\Omega) \Longrightarrow u = \bar{h}\,\overline{T}$

$u \perp \overline{H^2}(b\Omega) \Longrightarrow u = HT$

where $h, H \in H^2(b\Omega)$.

So $u = hT = \overline{H}\,\overline{T}$ on $b\Omega$.

Hmmm. $S_a^2 T = -\overline{L_a^2}\,\overline{T}$ on $b\Omega$.

Divide! $\qquad \dfrac{h}{S_a^2} = -\dfrac{\bar{H}}{L_a^2}$ on $b\Omega$.

$\dfrac{h}{S_a^2}$ analytic inside Ω and bndry values in $H^2(b\Omega)$.

So is $\dfrac{H}{L_a^2}$ because double pole of L_a^2 at $z=a$

makes $\dfrac{H}{L_a^2}$ have a zero at a.

Aha! These two fcns are both harmonic and agree on $b\Omega$. So they agree inside too!

$$\underbrace{\frac{h}{S_a^2}}_{\text{holo}} = \underbrace{\frac{\bar{H}}{\overline{L_a^2}}}_{\text{antiholo}} \quad \text{on } \Omega\,!$$

Both must be constant! [C-R eqns:

$f = u+iv$ holo and $\bar{f}$ holo $\Longrightarrow$ $\nabla u \equiv 0$, $\nabla v \equiv 0$.]

Contradiction: $\dfrac{H}{L_a^2} \equiv c$

no pole at a $\longrightarrow$ $H \equiv c\, L_a^2$ $\longleftarrow$ pole at a if $c \neq 0$.

So $c = 0$. And $u = \bar{H}\bar{T} \equiv 0$ too. ✓

Best proof: Harmonic fcns! Suppose $\psi \in C^\infty(b\Omega)$ and v harmonic extension of ψ to Ω. Know $v \in C^\infty(\bar{\Omega})$.

Claim $v = h + \bar{H}$, $h, H \in A^\infty(\Omega)$.

[Pf: $h = \frac{P(S_a v)}{S_a}$ $H = \frac{P(L_a v)}{L_a}$. Or use a real valued ψ. Get real valued v. Construct harm. conj. $\tilde{v}$, $h = v + i\tilde{v}$ holo. $v = \frac{1}{2}(h + \bar{h})$ ✓]

If $u \perp H^2$ and $\overline{H^2}$, then $u \perp \psi$.

$C^\infty(b\Omega)$ dense in $L^2(b\Omega) \Rightarrow u \equiv 0$.