Lecture 32 Domains with holes v.s. no holes

Last time: Best way to show that, if a fcn $u \in L^2(b\Omega)$ in a s.c. smooth domain is $\perp$ to $H^2(b\Omega)$ & $\perp \overline{H^2}(b\Omega)$, then $u \equiv 0$.

Best proof: Suppose $\psi \in C^\infty(b\Omega)$ and let v be the harmonic extension of ψ to Ω. Know $v \in C^\infty(\bar{\Omega})$. Furthermore, assume ψ real valued. So v is real valued too. Easy: v has a harmonic conjugate $\tilde{v}$ on Ω that is also in $C^\infty(\bar{\Omega})$.

[MA 530: $\begin{cases} v_x = \tilde{v}_y \\ v_y = -\tilde{v}_x \end{cases}$ $\qquad \nabla \tilde{v} \overset{\text{want}}{=} \begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

$$\nabla \varphi = \begin{pmatrix} \varphi_x \\ \varphi_y \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \Leftarrow \text{nec cond: } \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

MA 261: On a s.c. domain, cond is sufficient to cook up potential fcn φ for $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$.

Have it because $\begin{cases} v_{xy} = v_{yx} \\ v_{xx} = -v_{yy} \end{cases}$ ✓

$C^\infty(\bar{\Omega}) \to v_x = \tilde{v}_y$, $\quad v_y = -\tilde{v}_x$

So $\tilde{v}$ is in $C^\infty(\bar{\Omega})$ too.]

Write $h = v + i\tilde{v}$. So $v = \operatorname{Re} h = \frac{1}{2}(h + \bar{h})$.

Aha! $\psi = \frac{1}{2}h + \overline{\left(\frac{1}{2}h\right)}$ where $h \in H^2(b\Omega)$.

So $u \perp \Psi$. So $u \perp$ all real $C^\infty(b\Omega)$ fcns!

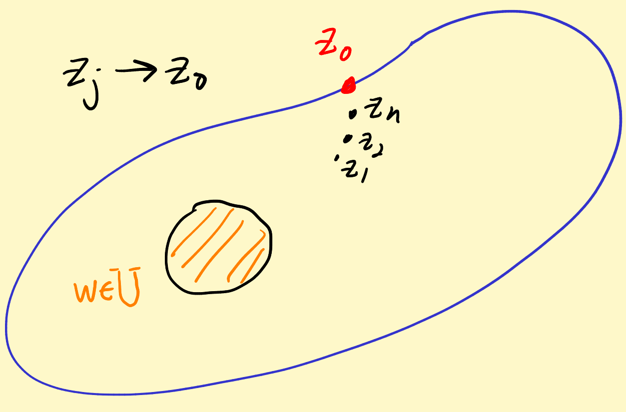
$C^\infty(b\Omega)$ dense in $L^2(b\Omega) \Rightarrow u = 0$ in $L^2(b\Omega)$.

Plan Prove all thms in detail for simply connected domains before moving on to domains with holes.

Last missing things: Have proved that $L(z,a) \neq 0$ if $z, a \in \Omega$ and $a \neq z$. Still haven't shown $L(z_0, a) \neq 0$ when $z_0 \in b\Omega$, $a \in \Omega$.

[This fact about $L(z_0,a)$ would also yield that $S(z_0,a) \neq 0$ for because $\overline{S_a} = \frac{1}{i} L_a T$ on $b\Omega$.]

Sketch of pf



show that $L(w, z_j)$ converge uniformly in $w \in \bar{U}$ to $L(w, z_0)$ as $j \to \infty$.

[Technical. Count derivatives in pf's. Not deep.]

MA 530: Hurwicz's thm: h_n non-vanishing analytic fcns that converge unif on compacts to h (analytic),

then $\quad$ A) $h \equiv 0$, or

B) h is <u>never</u> zero.

Get $L(w, z_0)$ is either $\equiv 0$ on $w \in \Omega$ or <u>never</u> zero.

How can $L(w, z_0) \equiv 0$ on Ω?

If $L(w, z_0) \equiv 0$, then $S(w, z_0) \equiv 0$ too! Must be impossible.

<u>Lemma</u>: $\underbrace{\left[\text{Complex linear span of } \{S_a : a \in \Omega\}\right]}$ $= \mathcal{S}$

is dense in $A^\infty(\Omega)$, meaning, given $h \in A^\infty(\Omega)$,

$\exists\ \sigma_j \in \mathcal{S}$ such that: Given $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$|\sigma_j - h| < \varepsilon \text{ on } \overline{\Omega} \quad \text{and}$$

$$|D^k \sigma_j - D^k h| < \varepsilon \quad \text{on } \overline{\Omega} \text{ for } k = 1, 2, \ldots, N$$

$$\left(\text{where } D^k = \frac{d^k}{dz^k}\right)$$

for all $j > m$.

Pf of Lemma: See Chap 9. [Still looking for better easier pf.]

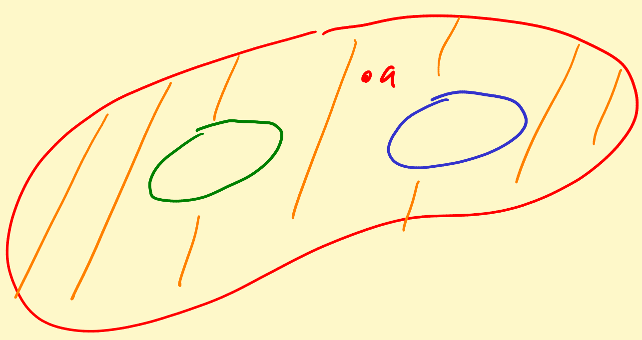
Lemma $\Rightarrow$ $S(w, z_0) \not\equiv 0$: $\quad S(z_0, w) = \overline{S(w, z_0)}$

Note: $1 \in \mathcal{S}$. $\quad \underset{\substack{\uparrow \\ \text{not zero} \\ \text{at } z_0.}}{1} \cong \sum_{1}^{N} c_j \underbrace{S(z_0, w_j)}_{\text{all zero}} \quad ⚡$.

What about domains with holes? Know that Riemann map $f_a \in C^\infty(\bar{\Omega})$ when Ω is smooth s.c..

History: Key reduction to domains with real analytic boundary

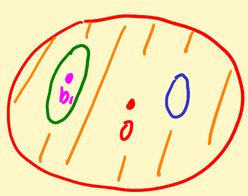
Ω



Step 1 Let $\tilde{\Omega}_1$ be s.c domain gotten by filling in holes of Ω.

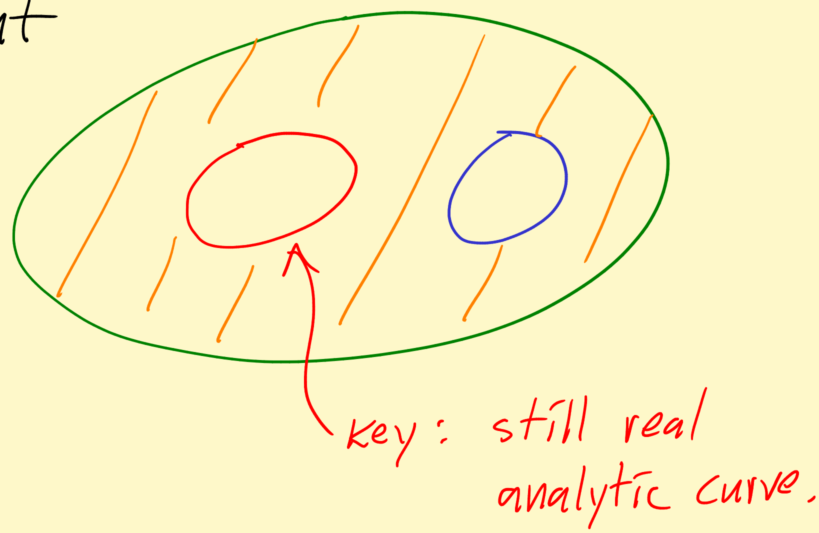
Step 2 Pick $a \in \Omega$. $\quad f_a : \tilde{\Omega}_1 \to D_1(0)$ Riemann map.

$\xrightarrow{f_a}$

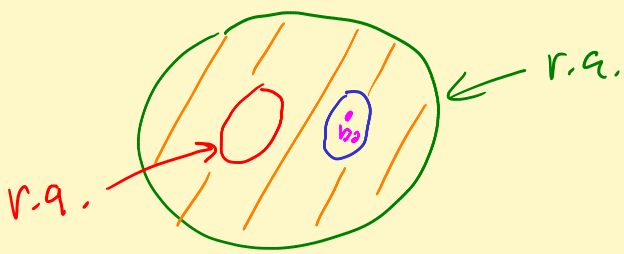


Step 3 Pick $b_1 \in$ hole. Do inversion $\frac{1}{z-b_1}$

Turns picture inside out



Repeat trick on outer curve.



Do one more time to make blue one r.a. too!

If more holes, keep repeating till last hole mapped to $C_1(0)$.