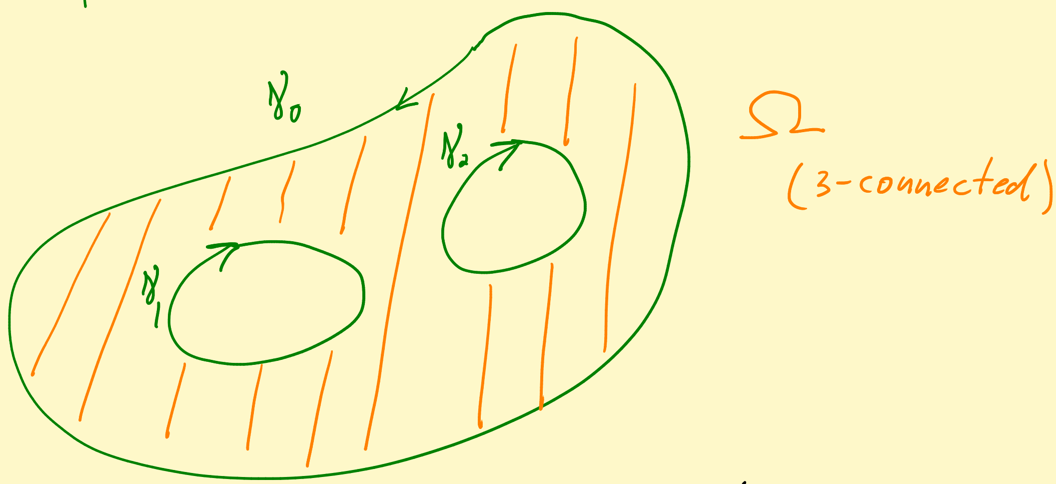
# Lecture 33 The Hopf lemma and consequences




Suppose $\Omega$ is a $C^\infty$-smooth $n$-connected domain bounded by $n$ nonintersecting $C^\infty$ Jordan curves.

Let $\gamma_0$ denote the outer boundary and $\gamma_j$, $j=1,\dots,n-1$ denote the inner boundaries.

**Two technicalities** 1) Not true that every harmonic fcn can be expressed as $h+\bar{H}$ where $h, H \in \mathcal{O}(\Omega)$.

2) Not true that $h \in \mathcal{O}(\Omega)$ has an analytic antiderivative on $\Omega$ for every $h \in \mathcal{O}(\Omega)$.

**Harmonic measure fcns** $\omega_j$ is the solution to the Dirichlet prob:

$$\begin{cases} \Delta \omega_j \equiv 0 & \text{on } \Omega \\ \omega_j\big|_{\gamma_j} \equiv 1 & \text{and } \omega_j\big|_{\gamma_k} \equiv 0,\ k \neq j. \end{cases}$$

Associated analytic fcns: $F_j' = 2\dfrac{\partial \omega_j}{\partial z}$ $\left(\Delta = 4\dfrac{\partial}{\partial \bar z}\dfrac{\partial}{\partial z}\right)$

Note: In spite of the notation, $F_j'$ is not the derivative of an analytic fcn on $\Omega$. But, locally, can write $h = w_j + iv$ ← harm conj

$$w_j = \frac{1}{2}(h + \bar{h})$$

$$\frac{\partial w_j}{\partial z} = \frac{1}{2} h'. \quad \text{So } h' = 2\frac{\partial w_j}{\partial z} = F_j' \text{ locally.}$$

Thm 1) Given harmonic fcn $u$ on $\Omega$, there are analytic fcns $h$ and $H$ on $\Omega$ and constants $c_1, \dots, c_{n-1}$ such that $u = h + \bar{H} + \sum_{j=1}^{n-1} c_j w_j$.

2) Given $h \in \mathcal{O}(\Omega)$, there is an analytic fcn $H$ on $\Omega$ and constants $\lambda_1, \dots, \lambda_{n-1}$ such that

$$h = H' + \sum_{j=1}^{n-1} \lambda_j F_j'.$$

Thm: $$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \lambda_{ij} F_i'(z) \overline{F_j'(w)}$$

Back to simply connected smooth bounded $\Omega$'s.

So nice that we've shown that the Riemann map $f_a \in C^\infty(\bar{\Omega})$! So $f_a'$ extends to $\bar{\Omega}$.

Hmmm. Can $f_a'(z_0)=0$ at a bndry pt $z_0$? (No!)

Hmmm. Maybe prove that $F=f_a^{-1}$ is also in $C^\infty(\overline{D_1(0)})$. That would rule out such zeroes. [We could do that now.]

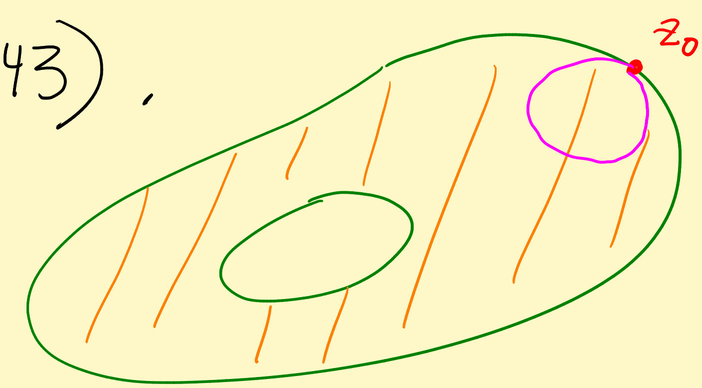
History. Use the "Hopf lemma".

Hopf lemma Suppose $u$ is a real valued non-constant harmonic fcn on our smooth domain that is in $C^1(\overline{\Omega})$ that assumes its maximum value on $\overline{\Omega}$ at a boundary pt $z_0$.

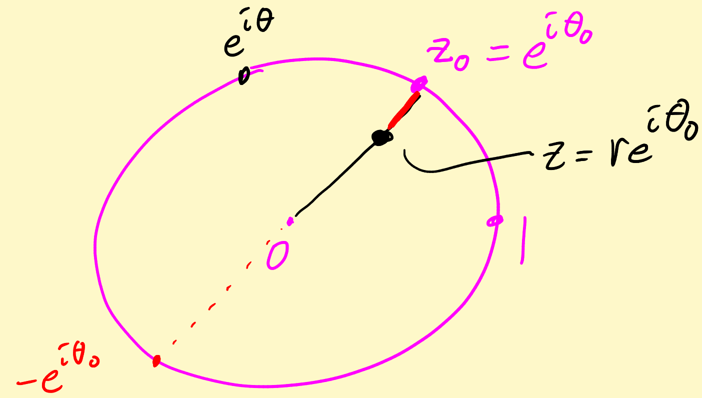
Then the (outward) normal derivative of $u$ at $z_0$

$$\frac{\partial u}{\partial n}(z_0) > 0.$$

Pf Simple consequence of Harnack's ineq (see Ahlfors p. 243).



Rescale, translate.




Replace $u$ by $\underbrace{u(z_0)}_{\text{max}} - u(z)$ $\Leftarrow$ $u$ non-const $\Rightarrow$ new $u$ is strictly $>0$ in $D_1(0) - \{z_0\}$, $u(z_0) = 0$

Harnick's use a simple estimate for Poisson kernel

$$P(z, \theta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2}$$

$$P(re^{i\theta_0}, \theta) = \frac{1}{2\pi} \frac{1 - r^2}{|e^{i\theta} - re^{i\theta_0}|^2}$$

Picture $(1-r)^2 \le \text{denom} \le (1+r)^2$. $\boxed{1 - r^2 = (1-r)(1+r)}$

So $$\frac{1}{2\pi} \frac{1-r^2}{(1+r)^2} \le P(re^{i\theta_0}, \theta) \le \frac{1}{2\pi} \frac{1-r^2}{(1-r)^2}$$

$$0 < \frac{1}{2\pi} \frac{1-r}{1+r} \le P(re^{i\theta_0}, \theta) \le \frac{1}{2\pi} \frac{1+r}{1-r}$$

Integrate $[0, 2\pi]$ against $u(e^{i\theta})$ :

$$\frac{1-r}{1+r} \cdot \underbrace{\left( \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) d\theta \right)}_{u(0)} \le u(re^{i\theta_0}) \le \frac{1+r}{1-r} u(0)$$

Aha! Positive harmonic fcns can't blow up worse than $\frac{1}{1-r}$ and can't $\searrow 0$ faster than $1-r$.

$$\frac{1-r}{1+r}\,u(0) \le u(re^{i\theta_0}) - \underbrace{u(e^{i\theta_0})}_{=0}$$

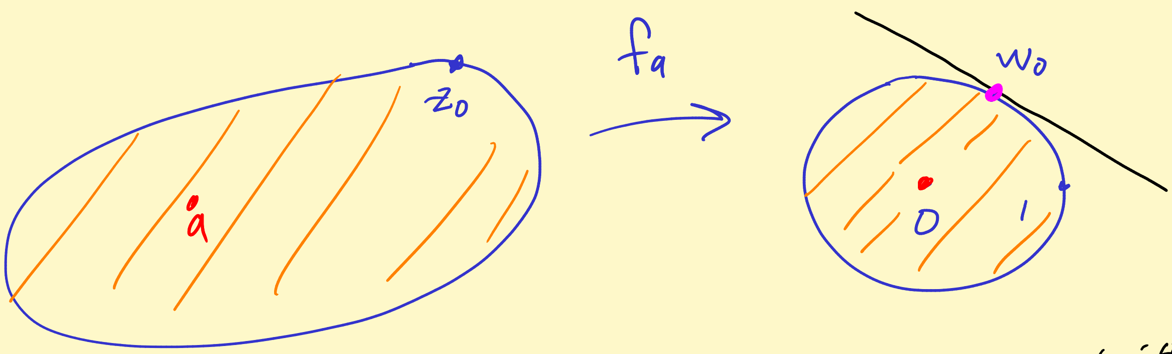
Divide by $1-r$

$$\frac{1}{1+r}\,u(0) \le \frac{u(re^{i\theta_0}) - u(e^{i\theta_0})}{1-r} \xrightarrow[r\nearrow 1]{} -\frac{\partial u}{\partial n}(e^{i\theta_0})$$

Thm: At max, $\frac{\partial}{\partial n} > 0$.

At min, $\frac{\partial}{\partial n} < 0$.

Next time: Riemann map




$u(z) = \mathrm{Re}\left(e^{-i\theta_0} z\right)$