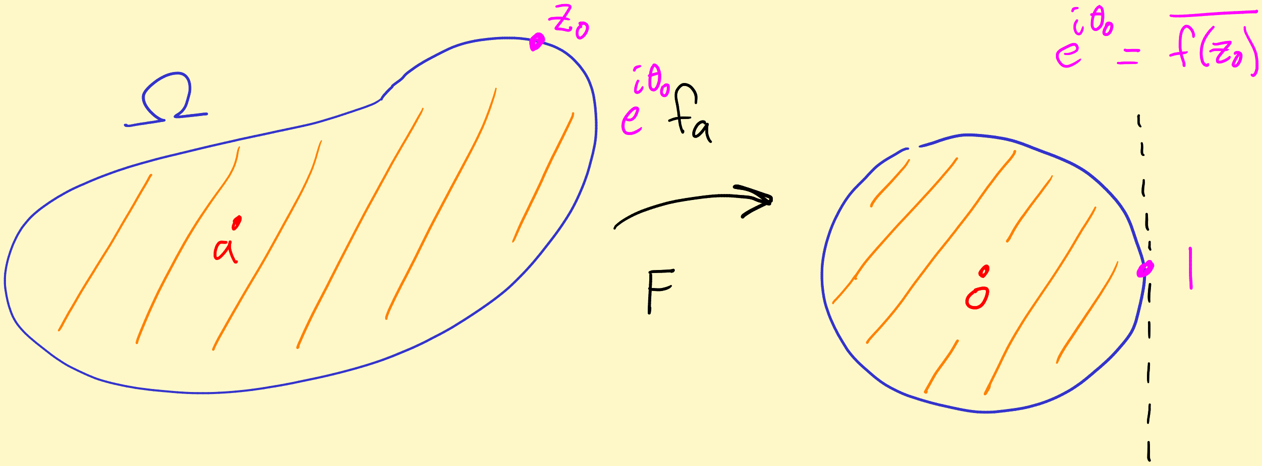
Lecture 34 Normal derivatives of harmonic fcns



$u(z) = \mathrm{Re}\, z$

$u(1) = \underset{D_1(0)}{\mathrm{Max}} \{u\}$

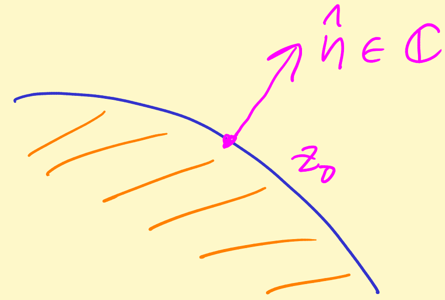
$u \circ F$ is harmonic on Ω, in $C^\infty(\overline{\Omega})$, non-constant, assumes absolute max on $\overline{\Omega}$ at z_0.

Hopf lemma $\Rightarrow \frac{\partial}{\partial n}(u \circ F)(z_0) > 0$.

This implies that $F'(z_0) \neq 0$. ["Conformality persists up to the boundary when boundaries are smooth".]

Why: $u = \mathrm{Re}\, z = \frac{1}{2}(z + \bar{z})$

$$u \circ F = \frac{1}{2}\left(F(z) + \overline{F(z)}\right)$$



$z(t) = z_0 + t\hat{n}$

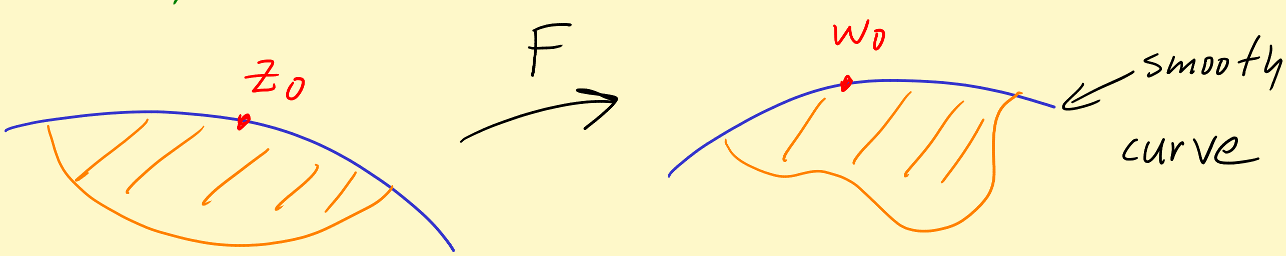
$$0 < \frac{\partial}{\partial n}(u \circ F) = \frac{d}{dt}\Big[\underbrace{u(F(z(t)))}_{\frac{1}{2}\left(F(z(t)) + \overline{F(z(t))}\right)}\Big]\Bigg|_{t=0}$$

$$= \frac{1}{2}F'(z(t))\,z'(t) + \frac{1}{2}\,\overline{F'(z(t))}\;\overline{z'(t)}\Bigg|_{t=0}$$

$$= \operatorname{Re}\,\underbrace{F'(z_0)\,\hat{n}}_{\text{can't be } 0.}$$

So $F'(z_0) \neq 0$. ✓

Remark Easy to "localize":



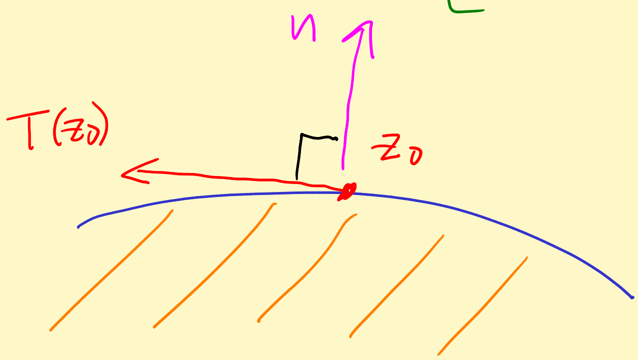
$F'(z_0) \neq 0$ is a local property for conformal maps that extend smoothly to the boundary.

Normal derivatives of harmonic fcns Suppose u is harmonic on a bounded smooth s.c. domain Ω in $C^\infty(\overline{\Omega})$. Say u is real valued. Then $u = \operatorname{Re} h$ where $h \in A^\infty(\Omega)$.

Normal derivative calculation

$$\frac{\partial}{\partial n} u(z_0) = \frac{d}{dt}\left[\tfrac{1}{2}\left(h(z(t)) + \overline{h(z(t))}\right]\right|_{t=0}$$

$$= \frac{1}{2}\left[h'(z_0)\,\hat{n} + \overline{h'(z_0)\,\hat{n}}\right]$$



$$\hat{n} = e^{-i\frac{\pi}{2}}\,T(z_0)$$

$$= -i\,T(z_0)$$

$$\left.\frac{\partial u}{\partial n}\right|_{z_0} = -\frac{i}{2} h'(z_0)\,T(z_0) + \frac{i}{2}\,\overline{h'(z_0)}\;\overline{T(z_0)}$$

Hmmm.

$$u = \frac{1}{2}\left[h + \bar{h}\right]$$

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left[h'(z) + 0\right]$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left[0 + \overline{h'(z)}\right]$$

<u>Fact</u>: $\frac{\partial u}{\partial n} = -i \underset{\text{holo}}{\underset{\uparrow}{\frac{\partial u}{\partial z}}} T(z) + i \underset{\text{antiholo}}{\underset{\uparrow}{\frac{\partial u}{\partial \bar{z}}}} \overline{T(z)}$

Fun with Dirichlet problem: Given bndry values φ, solution to the D. Prob was

$$u = \frac{P(\varphi S_a)}{S_a} + \overline{\left(\frac{P(\bar{\varphi} L_a)}{L_a}\right)}$$

<u>Classical Neumann problem</u> for Laplacian:

Given $\psi \in C^\infty(b\Omega)$ real valued,

find u harmonic on Ω such that

$$\frac{\partial u}{\partial n} = \psi.$$

<u>Important condition</u>: Must assume that

$$\int_{b\Omega} \psi \, ds = 0$$

for u to exist.

Why: Gauss' theorem [in $\mathbb{R}^2$: = Divergence thm = Green's thm = Stokes' thm]:

$$\int_{b\Omega} \underbrace{\frac{\partial u}{\partial n}}_{\psi} \, ds = \int_{b\Omega} \nabla u \cdot \hat{n} \, ds$$

$$= \iint_{\Omega} \underbrace{\mathrm{Div} \cdot \nabla u}_{\Delta u \equiv 0} \, dA = 0$$

Thm: $u = h + \overline{H}$ where

$$h' = S_a P(\psi / \overline{L_a})$$

$$H' = L_a P(\overline{\psi} / \overline{S_a})$$

and this shows that $u \in C^\infty(\overline{\Omega})$.

Note Pick a point $a \in \Omega$ and fix it. Recall that $S(z,a) \neq 0$ for all $z \in \overline{\Omega}$ and

$L(z,a) \neq 0$ for all $z \in \overline{\Omega} - \{a\}$.

Know $S_a(z) = S(z,a)$ and $L_a(z) = L(z,a)$ are smooth up to $b\Omega$. Know

$$L_a(z) = \underbrace{\frac{1}{2\pi(z-a)}}_{\text{simple pole}} + \underbrace{H_a(z)}_{\in A^\infty(\Omega)}$$

Also $\boxed{\overline{S_a} = \frac{1}{i} L_a T \quad \text{on } b\Omega}$

<u>Note</u> $H' = L_a \underbrace{P(\overline{\Psi}/\overline{S_a})}_{=0 \text{ at } z=a}$ ← Looks bad at $z=a$

, so it's not bad at a.

$$P(\overline{\Psi}/\overline{S_a})\Big|_{z=a} = \int_{b\Omega} S(a,z)\,\overline{\Psi(z)}/\overline{\underbrace{S_a(z)}_{\substack{S(z,a) \\ \| \\ \overline{S(a,z)}}}}\, ds_z$$

$$= \int_{b\Omega} \overline{\Psi}\, ds = 0 \checkmark$$

Singularity at $z=a$ is <u>removable</u>.

Game to find formula:

Say u harmonic solves Neumann problem.

Then $u = h + \overline{H}$ and

$$\frac{\partial u}{\partial n} = -i h' T + i \overline{H'} \, \overline{T}$$

Hmmn. $T = \frac{i \overline{S_a}}{L_a}$

$$\Psi = -i h' \frac{i \overline{S_a}}{L_a} + \underbrace{i \overline{H'} \, \overline{T}}_{\perp H^2(b\Omega)}$$

Hmmm. Divide by $\overline{S_a}$!

$$\frac{\Psi}{\overline{S_a}} = \underbrace{h'/L_a}_{\in H^2} + \underbrace{i \frac{\overline{H'}}{\overline{S_a}} \, \overline{T}}_{\perp H^2}$$

$P(\Psi/\overline{S_a})$ $\qquad$ $P^{\perp}(\Psi/\overline{S_a})$