Lecture 35 The classical Neumann problem in 2-D

Last time: If u is harmonic on a smooth domain Ω and smooth up to the boundary, then

$$\frac{\partial u}{\partial n} = -i\underset{\substack{\uparrow \\ \text{holo}}}{\frac{\partial u}{\partial z}} \cdot T + i \underset{\substack{\uparrow \\ \text{antiholo}}}{\frac{\partial u}{\partial \bar{z}}} \overline{T}$$

Remark: Showed for real valued u, so clearly true for complex valued harmonic fcns $u_1 + iu_2$ too.

Thm: If Ω is also simply connected and u is harmonic on Ω, then $u = h + \overline{H}$ for some $h, H \in \mathcal{O}(\Omega)$. [$h, H$ unique up to addition of constants.]

Thm: On a smooth simply connected domain, given $\Psi \in C^\infty(b\Omega)$ with $\int_{b\Omega} \Psi \, ds = 0$, the solution to the Neuman problem is $u = h + \overline{H}$ where

$$h' = S_a P(\Psi / \overline{L_a}) \leftarrow \text{def}^n$$

$$H' = L_a P(\overline{\Psi} / \overline{S_a})$$

This u is in $C^\infty(\bar{\Omega})$ and satisfies

$$\begin{cases} \Delta u \equiv 0 & \text{on } \Omega \\ \frac{\partial u}{\partial n} = \psi & \text{on } b\Omega \end{cases}$$

Pf: Define h and H as in statement of the thm.

Last time, Hmmm: If $u = h + \overline{H}$ and

$$\psi = \frac{\partial u}{\partial n} = -i\, h' \cdot T + i\, \overline{H' \cdot T}, \text{ then}$$

$$\psi = -i\, h' \cdot \left(\frac{i \bar{S}_a}{L_a} \right) + i\, \overline{H'}\, \overline{T}$$

$\boxed{\bar{S}_a = \frac{1}{i} L_a T}$

Divide by $\bar{S}_a$:

$$\frac{\psi}{\bar{S}_a} = \underbrace{h'/L_a + i\, \frac{\overline{H'}}{\bar{S}_a}\, \overline{T}}_{}$$

orthogonal decomp!

So $\frac{h'}{L_a} = P(\psi/\bar{S}_a)$. $\boxed{h' = L_a P(\psi/\bar{S}_a)}$ ✓

Recall the orthogonal decomp thm: $\boxed{v = g + \bar{G}\bar{T}}$

$v \in L^2(b\Omega)$. $g = Pv \in H^2(b\Omega)$. $\bar{G}\bar{T} = P^\perp v$

$$= \overline{P(\bar{v}\bar{T})}\, \bar{T}$$

Recall why $G = P(\bar{v}\bar{T})$: Multiply $\boxed{}$ by T
and take conjugate: $vT = gT + \bar{G}$

$$\bar{v}\bar{T} = G + \underbrace{\bar{g}\bar{T}}_{\perp H^2(b\Omega)}$$

$$G = P(\bar{v}\bar{T})$$

$$\boxed{\frac{\Psi}{\zeta_a} = h'/L_a + i\frac{\overline{H'}}{\zeta_a}\bar{T}}$$

So $$i\frac{\overline{H'}}{\zeta_a}\bar{T} = P^\perp\left(\frac{\Psi}{\zeta_a}\right) = \overline{P\left(\frac{\bar{\Psi}}{s_a}\bar{T}\right)}\;\bar{T}$$

Cancel $\bar{T}$. Take conjugate.

$$-i\frac{H'}{s_a} = P\left(\frac{\bar{\Psi}}{s_a}\bar{T}\right)$$

Aha! $\bar{\zeta}_a = \frac{1}{i}L_a T$

$T = i\bar{\zeta}_a/L_a$

$\bar{T} = -i s_a/\bar{L}_a$

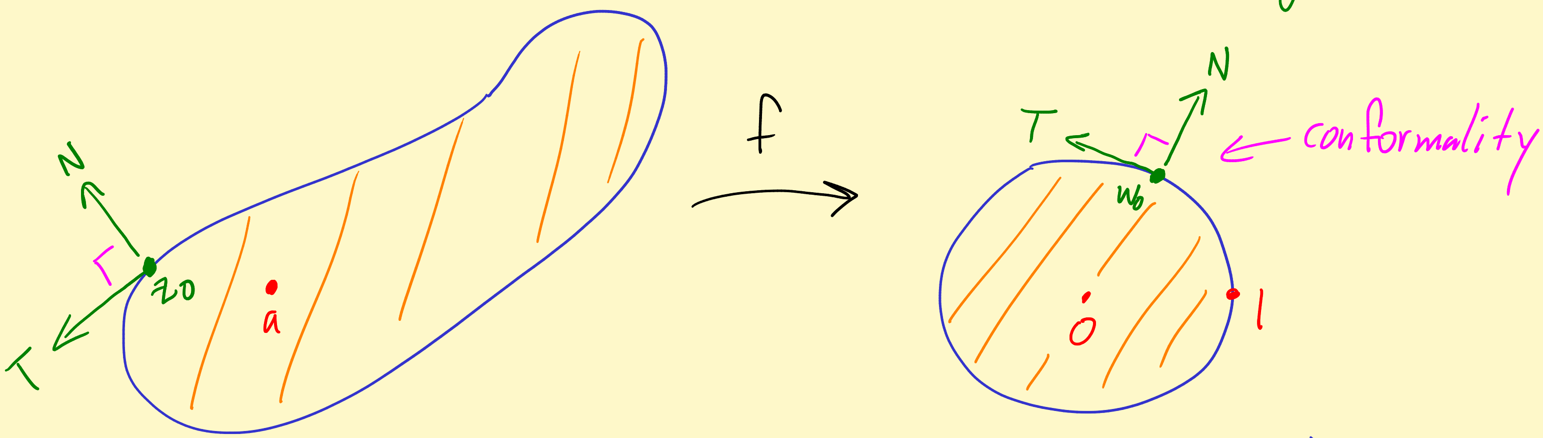
$$= P\left(\bar{\Psi}\cdot(-i/\bar{L}_a)\right)$$

$$\frac{H'}{s_a} = P(\bar{\Psi}/\bar{L}_a)$$

$$\boxed{H' = s_a P(\bar{\Psi}/\bar{L}_a)}$$

Now go backwards. Formulas work!

Hopf lemma $\Longrightarrow$ Important conformal mapping facts.



$b\Omega : z(t)$ $\qquad\qquad$ $C_1(0) : f(z(t))$

Standard sense:

$-iT = N \leftarrow$ outward normal

Conformality up to $b\Omega$ $\Longrightarrow$ $f(z(t))$ also standard sense!

Chain rule: $\quad \dfrac{d}{dt}\Big[f(z(t))\Big] = f'(z(t))\, z'(t)$

Hmmm: $\quad T(f(z_0)) = \dfrac{f'(z_0)\, T(z_0)}{|f'(z_0)\, T(z_0)|}$

$$\boxed{T(f(z_0)) = \frac{f'(z_0)}{|f'(z_0)|}\, T(z_0)}$$

Transformation formulas

Bergman projection and kernel

$$f : \Omega_1 \to \Omega_2$$

$$F = f^{-1} : \Omega_2 \to \Omega_1$$

$$\Lambda_1 : \ \varphi \in L^2(\Omega_2) \longmapsto f'(\varphi \circ f) \in L^2(\Omega_1)$$

Isometry

$$\Lambda_2 = \Lambda_1^{-1} : \ \psi \in L^2(\Omega_1) \longmapsto F'(\psi \circ F) \in L^2(\Omega_2)$$

Hilbert space fact: Isometries preserve inner product.

$$\langle \Lambda_1 \varphi, \psi \rangle_{L^2(\Omega_2)} = \langle \varphi, \Lambda_2 \psi \rangle_{L^2(\Omega_1)}$$

Berg proj transformation formula

$$\boxed{B_1\left(f' \cdot (\varphi \circ f) \right) = f' \cdot \left[(B_2 \varphi) \circ f \right]}$$

$$B_1 \circ \Lambda_1 = \Lambda_1 \circ B_2$$

Kernel functions transform too

$$\boxed{K_1(z,w) = f'(z)\, K_2(f(z), f(w))\, \overline{f'(w)}}$$

Szegő kernel version.

Claim $\lambda_1(\varphi) = \sqrt{f'} \cdot (\varphi \circ f)$ is a Hardy space isometry!

Simply connected case. (True for holes too.)

Know f' is non-vanishing on Ω because it is 1-1 holo. Hopf lemma $\Rightarrow$ f' non-vanishing on $b\Omega$ too. Ω s.c. $\Rightarrow$ f' has a holo square root on Ω. Pick one and call it $\sqrt{f'}$. [The other one is $-\sqrt{f'}$.]

Calculation:

$$\int_{b\Omega_2} u\,\bar{v}\,ds = \int_a^b u(f(z(t)))\,\overline{v(f(z(t)))}\left|\frac{d}{dt} f(z(t))\right| dt$$