Lecture 36 Transformation formulas

$$f : \Omega_1 \longrightarrow \Omega_2 \qquad F = f^{-1}$$

$b\Omega_1 : z(t) \qquad b\Omega_2 : f(z(t))$

$\boxed{ds = |z'(t)|\,dt} \qquad z'(t) \qquad \frac{d}{dt}\left[f(z(t))\right] = f'(z(t))\,z'(t)$

$$\langle u,v\rangle_2 = \int_{b\Omega_2} u\bar{v}\,ds = \int_a^b u(f(z(t)))\,\overline{v(f(z(t)))}\,\underbrace{|f'(z(t))|}_{\sqrt{f'}\,\overline{\sqrt{f'}}}\,\underbrace{|z'(t)|\,dt}_{ds \text{ on } b\Omega_1}$$

$$= \int_a^b \sqrt{f'(z(t))}\,u(f(z(t)))\,\overline{\sqrt{f'(z(t))}\,v(f(z(t)))}\,|z'(t)|\,dt$$

$$= \int_{b\Omega_1} \sqrt{f'}\,(u\circ f)\,\overline{\sqrt{f'}\,(v\circ f)}\,ds$$

$$\langle u,v\rangle_{b\Omega_2} = \langle \lambda_1(u), \lambda_1(v)\rangle_{b\Omega_1}$$

Thm: $\lambda_1 : u \longmapsto \sqrt{f'}\,(u\circ f)$ is an isometry of $L^2(b\Omega_2) \longleftrightarrow L^2(b\Omega_1)$

Pf: Let $u=V$ above. The inverse of λ_1 is

$$\lambda_2 : V \longmapsto \sqrt{F'}\,(V\circ F)$$

if we choose $\sqrt{F'}$ to the one such that

$$\sqrt{F'(f(z))} = \frac{1}{\sqrt{f'(z)}}. \qquad \underline{\text{Pf}}: \text{Inverse fcn thm (Chain rule)}$$

Remark: Calculation above showed that

$$\langle u, v \rangle_{b\Omega_2} = \langle \lambda_1(u), \lambda_1(v) \rangle_{b\Omega_1}$$

also preserves the inner products.

Always follows from fact that λ_1 is an isometry via the "polarization identity":

$$\langle u, v \rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 \right) - \frac{i}{4}\left(\|u+iv\|^2 - \|u-iv\|^2 \right)$$

More games $\quad \langle u, v \rangle_2 = \langle \lambda_1(u), \lambda_1(v) \rangle_1$

$v = \lambda_2(V)$

$$\boxed{\langle u, \lambda_2(V) \rangle_2 = \langle \lambda_1(u), V \rangle_1}$$

Kernel fcns transformation Play around replacing u, v by S_a, S_b.

$$\langle h, \lambda_1(S_a) \rangle_{b\Omega_2} = \langle \underbrace{\lambda_2(h)}, S_a \rangle_{b\Omega_1}$$

$h \in H^2(b\Omega_2)$ $\qquad \in H^2(b\Omega_1)$

$$= \lambda_2(h)\Big|_a$$

$$= \sqrt{F'(a)}\; h(F(a))$$

Hmmm. Another obvious fcn does same thing when paired with h:

It is $c\, S_{F(a)}$ where $c = \overline{\sqrt{F'(a)}}$.

So $\langle h, \lambda_1(S_a) \rangle_{b\Omega_1} = \langle h, \overline{\sqrt{F'(a)}}\, S_{F(a)} \rangle$

for all $h \in H^2(b\Omega_1)$. So

$$\lambda_1(S_a) - \overline{\sqrt{F'(a)}}\, S_{F(a)} \perp H^2(b\Omega_1)$$

and in $H^2(b\Omega_1)$, so $\equiv 0$. So

$$\boxed{\sqrt{f'(z)}\; S_2(f(z), a) = S_1(z, F(a))\, \overline{\sqrt{F'(a)}}}$$

Let $a = f(w)$ and use chain rule to get

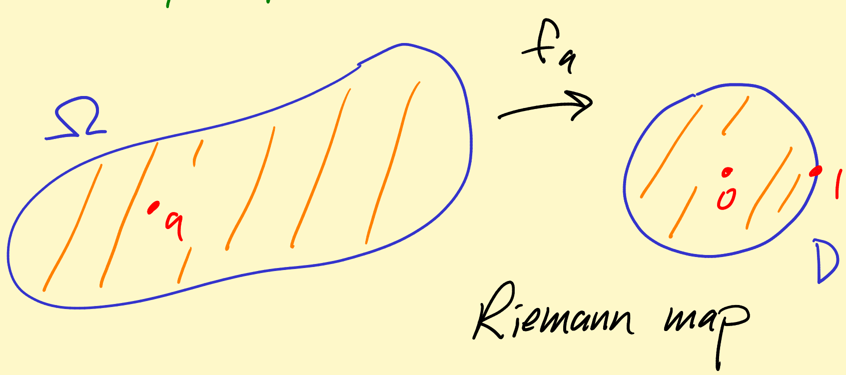
$$\boxed{\sqrt{f'(z)}\, S_2(f(z), f(w)) \sqrt{\overline{f'(w)}} = S_1(z,w)}$$

Remark Same proofs work for Λ_1, Λ_2 in Bergman space.

Skipped part of thm λ_1 is an isometric isomorphism of L^2 spaces that preserves the Hardy spaces.

$\lambda_2 = \lambda_1^{-1}$ does same thing. So λ_1 is also an isomet. iso. of Hardy spaces.

Fun thing to do



$$S_D(z,w) = \frac{1}{2\pi} \frac{1}{1 - z\bar{w}}$$

$$\sqrt{f_a'(z)}\, \underbrace{S_D(z, 0)}_{1/2\pi} = S_\Omega(z, \underbrace{F(0)}_{a}) \sqrt{\overline{F'(0)}}$$

$$\sqrt{f_a'(z)} = \underbrace{2\pi \sqrt{\overline{F'(0)}}}_{c}\, S_\Omega(z,a)$$

Hmmm. Know $f_a'(a) \in \mathbb{R}^+$. ← Smart to choose $\sqrt{f_a'}$ so this is > 0.

Then $2\pi \sqrt{\sqrt{F'(0)}}$ is also > 0.

Play around: See $c = \sqrt{\frac{2\pi}{S(a,a)}}$ by setting $z=a$.

Finally, square it

$$\boxed{f_a'(z) = 2\pi \frac{S(z,a)^2}{S(a,a)}}$$

← Let $z=a$

Fun fact : $f_a'(a) = 2\pi\, S(a,a)$

Remark: Also get an alternate proof that

$$K(z,w) = 4\pi\, S(z,w)^2$$

in s.c. case.

Interesting thing Transformation formulas for Szegő things works same if domains have holes because f' has an analytic square root.

Consequently: We can map domains with holes to domains with real analytic boundaries to greatly simplify proofs.