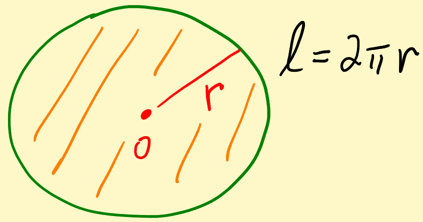
Lecture 37 The Isoperimetric inequality



$A = \pi r^2$

$$A = \pi r^2 = \pi \left(\frac{\ell}{2\pi}\right)^2 = \frac{\ell^2}{4\pi} \leftarrow \text{disc}$$

$$A \le \frac{\ell^2}{4\pi} \leftarrow \text{general } \Omega$$

and if $=$, the $\Omega = \text{disc}$.

Assume Ω is a smooth s.c. domain bounded by a real analytic curve. Know via Schwarz reflection that Riemann map $f: \Omega \longrightarrow D_1(0)$ extends analytically to an open set containing $\overline{\Omega}$. Let $F = f^{-1}$.

Note: F extends holo to open set containing $\overline{D_1(0)}$.

Pf: (T. Carleman) [Dragan Vukotić, "The isoperimetric ineq and a thm of Hardy & Littlewood," Math Monthly 110, 2003.]

Clever trick: Expand $\sqrt{F'}$ in power series and compare L^2-norms.

Facts: z^n are orthogonal in both the Bergman and Hardy spaces for $D_1(0)$.

Orthonormal basis: Bergman: $\left\{ \frac{z^n}{\left(\sqrt{\frac{\pi}{n+1}}\right)} \right\}_{n=0}^{\infty}$

Hardy: $\left\{ \frac{z^n}{\sqrt{2\pi}} \right\}_{n=0}^{\infty}$

On disc. Power series are Hilbert space $\perp$ decomopositions!

Step 1: $\sqrt{F'} = \sum_{n=0}^{\infty} a_n z^n$

$$A(\Omega) = \iint_{D_1(0)} \underbrace{|F'|^2}_{} \, dA$$

C-R eqns $\Rightarrow$ $|F'|^2 = |\text{Jacobian det}_{\mathbb{R}^2 \to \mathbb{R}^2}|$

$$= \iint_{D_1(0)} \left|\sqrt{F'}\right|^4 \, dA$$

$$= \left\langle \left(\sqrt{F'}\right)^2, \left(\sqrt{F'}\right)^2 \right\rangle_{D_1(0)} = \left\|\left(\sqrt{F'}\right)^2\right\|^2$$

Hmmm. $\sqrt{F'} = \sum_0^{\infty} a_n z^n$

$$\left(\sqrt{F'}\right)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) z^n \cdot \frac{\left(\sqrt{\frac{\pi}{n+1}}\right)}{\left(\sqrt{\frac{\pi}{n+1}}\right)}$$

orthog sum!

So $A(\Omega) = \langle \; , \; \rangle = \sum_{n=0}^{\infty} \underbrace{\frac{\pi}{n+1}}_{!} \left| \sum_{k=0}^{n} \underbrace{a_k a_{n-k}}_{b_k} \right|^2$

Miracle. Cauchy-Schwarz ineq is the core of pf!

$$\left|\sum_{k=0}^{n} b_k\right|^2 = \left|(b_0, b_1, \ldots, b_n)\cdot(1,1,\ldots,1)\right|^2$$

$$\leq \left[\left(\sum_{k=0}^{n}|b_k|^2\right)^{1/2}\cdot\underbrace{\left(\sum_{k=0}^{n}1\right)^{1/2}}_{\sqrt{n+1}}\right]^2$$

$$\leq (n+1)\sum_{k=0}^{n}|b_k|^2$$

So $A(\Omega) \leq \pi \underbrace{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}|a_k|^2|a_{n-k}|^2\right)}$

Aha! $= \left(\sum_{n=0}^{\infty}|a_n|^2\right)^2$

So far: $A(\Omega) \leq \pi\left(\sum_{n=0}^{\infty}|a_n|^2\right)^2$

Next: $\ell(\Omega) = \int_{C_1(0)} |F'|\, ds$ ← $F(e^{it})$ parametrizes $b\Omega$.

$$= \left\langle \sqrt{F'}, \sqrt{F'} \right\rangle_{C_1(0)} = \|\sqrt{F'}\|^2_{C_1(0)}$$

$$\sqrt{F'} = \sum_{n=0}^{\infty} a_n z^n = \sqrt{2\pi}\sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{2\pi}}$$

$$\ell(\Omega) = 2\pi \sum_{n=0}^{\infty} |a_n|^2$$

Done! $\quad A(\Omega) \leq \pi \Big(\underbrace{\sum_{n=0}^{\infty} |a_n|^2}_{\frac{\ell(\Omega)}{2\pi}} \Big)^2$ ✓

What about equality? Cauchy-Schwarz ineq is unequal only when the vectors $(b_0, \dots, b_n)$ and $(1, \dots 1)$ are not pointing in same $(\pm)$ directions.

Equality: $(b_0, \dots, b_n) = c(1, \dots 1)$.

Play around. Use induction. Get $a_n = \lambda^n a_0$ for some const $\lambda \in \mathbb{C}$ and cousequently

$$F(z) = \frac{1}{1 - \lambda z} \leftarrow \text{Linear fractional transf!}$$

So Ω is either a half-plane ← not! or a disc.

Similar Fourier series proof. Not surprising because Hardy space on $D_1(0)$ was invented to study connections of boundary behaviour of analytic fcns and Fourier series.

$$\left(\sum_{n=-\infty}^{\infty} c_n e^{in\theta}\right) = \left(\sum_{n=0}^{\infty} c_n e^{in\theta}\right) + \sum_{n=1}^{\infty} c_{-n} e^{-in\theta}$$

$$= \left(\sum_{n=0}^{\infty} c_n z^n\right)\Bigg|_{z=e^{i\theta}} + \left(\sum_{n=1}^{\infty} c_{-n} z^{-n}\right)\Bigg|_{z=e^{i\theta}}$$